

All homogeneous $N = 2$ M-theory truncations with supersymmetric AdS_4 vacua

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Abstract

We study consistent truncations of M-theory to gauged $N = 2$ supergravity in four dimensions, based on a large class of $\text{SU}(3)$ -structures in seven dimensions. We show that the gauging involves isometries of the vector multiplet scalar manifold as well as the Heisenberg algebra and a special isometry of the hyperscalar manifold. As a result, non-abelian gauge groups and new non-trivial scalar potentials are generated. Then we specialize to all homogeneous $\text{SU}(3)$ -structures supporting supersymmetric AdS_4 vacua. These are the Stiefel manifold $V_{5,2}$, the Aloff–Wallach spaces $N(k, l)$, the seven-sphere (seen as $\text{SU}(4)/\text{SU}(3)$ or $\text{Sp}(2)/\text{Sp}(1)$) and the M^{110} and Q^{111} coset spaces. For each of these cases, we describe in detail the $N = 2$ model and discuss its peculiarities.

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1 Introduction

A fruitful spin-off of the AdS/CFT-motivated classification of supersymmetric string and M-theory backgrounds containing AdS factors has been the establishment of consistent truncations of ten- and eleven-dimensional supergravity down to lower-dimensional gauged supergravities with AdS vacua [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Consistent truncations in this context provide well-defined interacting theories for a finite number of modes on AdS, whose dynamics is guaranteed to also solve the higher-dimensional theory. Consistently truncated theories are thus very useful to economically describe dual field-theoretic phenomena away from the superconformal point, like RG evolution (see e.g. [14, 15, 16, 17, 18]), finite temperature and chemical potential behaviour (e.g. [19, 20, 21, 22, 23, 24, 25]), or deformations into non-relativistic regimes (e.g. [26, 27, 28, 29]). The interest in consistent truncations is of course not limited to a holographic context: well-known early results include the maximally supersymmetric, spherical truncations to four dimensions of [30] (recently updated in [31]); to seven dimensions of [32, 33]; and partial results to five dimensions in e.g. [34].

This paper is a step forward in the programme of classifying lower-dimensional supergravity theories that uplift to eleven-dimensional (11D) supergravity. We establish all possible 4D, $N = 2$ gauged supergravities that arise from left-invariant consistent truncation of 11D supergravity on 7D coset manifolds $M_7 = G/H$, and that contain supersymmetric AdS₄ vacua. These vacua can preserve either the full $N = 2$ supersymmetry of the action, or spontaneously break it to $N = 1$. By left-invariant (LI) truncation we mean that the Kaluza–Klein tower of M-theory on $M_7 = G/H$ is truncated to the sector invariant under the left-action of the group G . This ensures consistency by a symmetry principle: G -neutral modes cannot source G -charged fields.

The requirement that the truncated theory is $N = 2$ supergravity implies that the coset space admits a left-invariant SU(3)-structure. This condition, together with the further requirement on the existence of a supersymmetric AdS₄ solution, specifies the possible internal manifolds to the list in table 1. Indeed, we have shown that, if homogeneity of the internal space is assumed, the most general supersymmetric M-theory background containing AdS₄ is necessarily of Freund–Rubin type. In other words, homogeneity of M_7 rules out warp factors, trivially, and also internal fluxes of the 11D four-form G_4 . Homogeneous supersymmetric AdS₄ \times M_7 Freund–Rubin solutions to 11D supergravity were classified long ago [35, 36], and the internal manifolds M_7 that also admit a left-invariant SU(3)-structure are precisely the Stiefel manifold $V_{5,2}$, the Aloff–Wallach spaces $N(k, l)$, the seven-sphere (seen as SU(4)/SU(3) or Sp(2)/Sp(1)) and the M^{110} and Q^{111} coset spaces listed in table 1.

M_7	G	H	B_6	N AdS ₄
S^7	SU(4)	SU(3)	\mathbb{CP}^3	2
M^{110}	$SU(3) \times SU(2)$	$SU(2) \times U(1)$	$\mathbb{CP}^2 \times \mathbb{CP}^1$	2
Q^{111}	$SU(2)^3$	$U(1)^2$	$\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$	2
$V_{5,2}$	SO(5)	SO(3)	$Gr(5, 2)$	2
S^7	Sp(2)	Sp(1)	\mathbb{CP}^3	1 and 2
$N(1, 1)$	SU(3)	U(1)	$F(1, 2)$	1 and 2
$N(k, l)$	SU(3)	U(1)	$F(1, 2)$	1
$N(1, -1)$	SU(3)	U(1)	$F(1, 2)$	1

Table 1: All coset manifolds $M_7 = G/H$ admitting a G -invariant SU(3)-structure and supporting a supersymmetric, homogeneous AdS₄ \times M_7 compactification of 11D supergravity. These are also all compact 7D coset manifolds that admit a G -invariant SU(3)-structure and do not have U(1) factors. The fourth column denotes the 6D base B_6 of the 7D manifold (cf. section 2.1); here, Gr denotes the Grassmannian and F the flag manifold. The last column gives the amount of supersymmetry of the AdS₄ solutions within our $N = 2$ models.

The list of [35, 36] actually comprises other homogeneous supersymmetric solutions of 11D supergravity to which a LI consistent truncation can be associated, but for these instances the latter is not based on an SU(3)-structure (hence does not lead to $N = 2$ supergravity) and has already been covered by previous work. Indeed, going through the list of [35, 36], we find that $SO(5)/SO(3)_{\max}$ and $Spin(7)/G_2$ admit no new LI truncation to $N = 1$ supergravity beyond the universal weak G_2 truncation of [5].¹ All $N = 4$ and $N = 3$ LI cases were covered in the universal tri-Sasakian truncation of [13]. Moreover, Freund–Rubin backgrounds with $4 \leq N \leq 8$ can only arise when M_7 is (an orbifold of) the round S^7 and, as in the $N = 8$ truncation of [30], the associated Kaluza–Klein ansätze necessarily involve inhomogeneous deformations. We can thus conclude that the consistent truncations in this paper, along with those of [5, 13] are, up to subtruncation, all the possible LI consistent truncations of 11D supergravity on homogeneous 7D manifolds with supersymmetric AdS₄ vacua. Note that this conclusion crucially relies on left-invariance: we can make no claims about further possible consistent truncations which, like the $N = 8$ truncation on S^7 , deform inhomogeneously the internal space.

Our results actually reach further generality in the following way. Rather than analysing the possible $N = 2$ theories associated to the cosets of table 1 on a case-by-case basis, we instead introduce a family of manifolds with SU(3)-structure that contains the cosets as particular cases, and perform the dimensional reduction on this larger general class.

¹Only recently have compact, homogeneous G -invariant G_2 -structure seven-folds been classified in the mathematics literature [37]. The results agree with [35, 36].

In order to do this, we borrow some usual techniques from the flux compactification literature. Namely, we will proceed similarly to the familiar, and closely related, compactifications of type II supergravity on $SU(3)$ -structure six-folds, which themselves generalise the Calabi–Yau compactifications with background fluxes (see e.g. [38] for a review). Following [39, 40, 41, 42], we assume that the 7D manifold is equipped with a set of forms, typically not harmonic but instead postulated to obey certain algebraic and differential conditions. We then use this set of forms to expand the supergravity three-form and the $SU(3)$ -structure forms, whose corresponding moduli we show to be governed by special Kähler geometry. The resulting truncated theory is $N = 2$ supergravity, coupled to vector and hypermultiplets, with a rich family of both electric and magnetic gaugings inherited from the intrinsic torsion of the $SU(3)$ -structure and from the four-form flux. We find that in general these gaugings involve both isometries of the vector multiplet and of the hypermultiplet scalar manifolds, leading to a non-abelian gauge algebra, and to new, non-trivial scalar potentials. Examples of M-theory reductions to $N = 2$ supergravity with charged vector multiplet scalars are known [43, 44], and we extend these results to a larger class of gaugings, including a special isometry (also studied in [44]) and the Heisenberg algebra of the hyperscalar manifold.

An important feature of our reduction scheme is that the algebraic restrictions on the expansion forms will be imposed here with no integration over the internal manifold. This stronger requirement ensures that reduction on our general class of manifolds can in many cases be performed not only at the level of the action but also at the level of the equations of motion. The latter feature implies consistency of the truncation. The benefits of this approach are manifold. It not only allows us to simultaneously manage the reduction on the cosets of table 1 in a systematic and unified way, but it also allows to include further consistent truncations on a larger class of internal spaces not any longer required to be homogeneous. Our general class of internal spaces contains, for example, any Sasaki–Einstein (SE_7) space. Accordingly, our 4D $N = 2$ family of supergravities includes the universal truncation of 11D supergravity on SE_7 [5]. It also includes the largest $N = 2$ subtruncation of the universal $N = 4$ truncation on tri-Sasakian manifolds [13, 45]. Moreover, our family of $N = 2$ theories also contains the models arising from consistent truncation of type IIA on nearly-Kähler and homogeneous $SU(3)$ -structure six-folds [6, 46], in the limit of vanishing Romans mass. As an aside, it is interesting to note that our family of $N = 2$ gaugings does also allow for a further extension, no longer immediately compatible with a consistent 11D uplift, by an additional parameter that in type IIA admits a straightforward interpretation as a Romans mass.

Contact between our general family of $SU(3)$ -structure manifolds and the cosets of table 1 is achieved when the postulated set of forms is taken to be the set of LI forms on each coset. In fact, the assumptions on the forms defined on our general class of manifolds are just abstractions of those defined by the LI forms. Obtaining the particular theory arising for each coset from our general family of $N = 2$ gauged theories is just a matter of parameter-fixing.

M_7	n_V	n_H	VM s.m.	\mathcal{F}	HM s.m.	\mathcal{G}
$S^7 = \frac{\text{SU}(4)}{\text{SU}(3)}$	1	1	$\frac{\text{SU}(1,1)}{\text{U}(1)}$	$-\frac{(X^1)^3}{X^0}$	$\frac{\text{SU}(2,1)}{\text{S}(\text{U}(2) \times \text{U}(1))}$	—
M^{110}	2	1	$\left(\frac{\text{SU}(1,1)}{\text{U}(1)}\right)^2$	$-\frac{(X^1)^2 X^2}{X^0}$	$\frac{\text{SU}(2,1)}{\text{S}(\text{U}(2) \times \text{U}(1))}$	—
Q^{111}	3	1	$\left(\frac{\text{SU}(1,1)}{\text{U}(1)}\right)^3$	$-\frac{X^1 X^2 X^3}{X^0}$	$\frac{\text{SU}(2,1)}{\text{S}(\text{U}(2) \times \text{U}(1))}$	—
$V_{5,2}$	1	2	$\frac{\text{SU}(1,1)}{\text{U}(1)}$	$-\frac{(X^1)^3}{X^0}$	$\frac{\text{G}_{2(2)}}{\text{SO}(4)}$	$-\frac{(Z^1)^3}{3\sqrt{3}Z^0}$
$S^7 = \frac{\text{Sp}(2)}{\text{Sp}(1)}$	2	2	$\left(\frac{\text{SU}(1,1)}{\text{U}(1)}\right)^2$	$-\frac{(X^1)^2 X^2}{X^0}$	$\frac{\text{SO}(4,2)}{\text{SO}(4) \times \text{SO}(2)}$	$\frac{(Z^0)^2 - (Z^1)^2}{2i}$
$N(1, 1)$	3	2	$\left(\frac{\text{SU}(1,1)}{\text{U}(1)}\right)^3$	$-\frac{X^1 X^2 X^3}{X^0}$	$\frac{\text{SO}(4,2)}{\text{SO}(4) \times \text{SO}(2)}$	$\frac{(Z^0)^2 - (Z^1)^2}{2i}$
$N(k, l)$	3	1	$\left(\frac{\text{SU}(1,1)}{\text{U}(1)}\right)^3$	$-\frac{X^1 X^2 X^3}{X^0}$	$\frac{\text{SU}(2,1)}{\text{S}(\text{U}(2) \times \text{U}(1))}$	—
$N(1, -1)$	5	1	$\frac{\text{SU}(1,1)}{\text{U}(1)} \times \frac{\text{SO}(4,2)}{\text{SO}(4) \times \text{SO}(2)}$	$\frac{X^3[-X^1 X^2 + (X^4)^2 + (X^5)^2]}{X^0}$	$\frac{\text{SU}(2,1)}{\text{S}(\text{U}(2) \times \text{U}(1))}$	—

Table 2: Summary of some relevant features of the $N = 2$ gauged supergravities obtained from LI consistent truncation of 11D supergravity on M_7 . The table shows the number of vector multiplets (n_V) and hypermultiplets (n_H), and the respective scalar manifolds (VM s.m. and HM s.m.) with the associated special Kähler prepotentials (\mathcal{F} and \mathcal{G}).

The cosets of table 1 admit a LI moduli space of $\text{SU}(3)$ -structures, where two significant ones characterized by specific torsion classes may arise, defining the preserved supersymmetry of the corresponding AdS vacuum. Either one or both structures may be present for each coset. These special LI $\text{SU}(3)$ -structures are, on the one hand, the well-known SE_7 structure and, on the other hand, a certain, $\text{U}(1)$ -invariant, type of weak G_2 structure, which can be regarded as a circle fibration over (the vanishing Romans mass limit of) the class of half-flat six-folds discussed in [47, 48, 49]. Accordingly, although both types of manifolds give rise to $N = 2$ supergravity actions, the former preserve $N = 2$ in the vacuum while the latter spontaneously breaks $N = 2 \rightarrow N = 1$.² Referring to table 1, the first six cosets can be endowed with a SE_7 structure, and in fact exhaust the list of all possible homogeneous SE_7 spaces, see e.g. [52]. The last four cosets admit the second type of structure. The two cosets admitting both types of $\text{SU}(3)$ -structures in fact admit a LI tri-Sasaki structure, which allows both for a SE_7 and (when squashed) for a distinct weak G_2 structure.

We have summarized in table 2 the main features of the $N = 2$ effective theories arising from LI consistent truncation on the cosets of table 1. As noted in [13], the LI truncation

²See [50, 51] for a recent general discussion of spontaneous supersymmetry breaking in gauged $N = 2$ supergravity.

on $SU(4)/SU(3)$ coincides with the universal SE_7 truncation of [5], which comprises one vector multiplet and one hypermultiplet. The truncations on M^{110} and Q^{111} merely add one and two Betti vector multiplets, respectively, to the universal SE_7 truncation, while that on the Stiefel manifold $V_{5,2}$ includes an additional hypermultiplet. The $N = 2$ models on $Sp(2)/Sp(1)$ and $N(1,1)$ that we present here are actually subtruncations of the full $N = 4$ tri-Sasakian truncation of [13] obtained by imposing a certain \mathbb{Z}_2 symmetry, but were not discussed in that paper. In particular, the $N = 2$ theory based on $Sp(2)/Sp(1)$ is, like its $N = 4$ parent, universally valid for any tri-Sasakian space. In the symplectic frame specified by the truncation, the gauging is both electric and magnetic across all models (a summary is given in tables 3, 4 below). The gauge group is always abelian and only the hyperscalars pick up charges, except for the Aloff–Wallach space $N(1, -1)$, where there is also an interesting, non-abelian gauging in the vector multiplet sector. It should be noted that some partial results about these 4D truncations have already appeared in the literature: in [42, 53] for the $V_{5,2}$ and the $N(k, l)$ models, and in [53, 54] for the M^{110} and Q^{111} models. Reference [42] also discussed some general features of the dimensional reduction of M-theory on $SU(3)$ -structure seven-folds. However, none of these references specifies both the full overarching $N = 2$ theory and the details of each model as we do in the present work.

The structure of the rest of the paper is as follows. In section 2 we describe the two canonical $SU(3)$ -structures supported by the cosets of table 1, and introduce the general class of $SU(3)$ -structure manifolds compatible with these, but not limited to them. We study its moduli space and show it to be governed by special geometry. The reduction on our general class of spaces is performed in section 3, where the resulting general family of 4D theories is also presented. This is shown to be compatible with gauged $N = 2$ supergravity in section 4. In section 5 we retrieve the explicit models of table 2 from our general reduction, and we further discuss their salient features. Section 6 concludes with an outlook. The appendix contains the proof that homogeneous supersymmetric AdS_4 solutions to 11D supergravity are of Freund–Rubin type (appendix A), an algorithm to compute the prepotential on the LI almost complex structure moduli space of our internal manifolds (appendix B), as well as technical details on the dimensional reduction (appendix C), and matching with $N = 2$ supergravity (appendix D). We have also included a self-contained description of the LI geometry on the relevant cosets (appendix E). Finally, we give (appendix F) the spectrum of each coset model at every supersymmetric critical point, and its arrangement into Osp multiplets.

2 SU(3)-structures and their moduli space

2.1 SU(3)-structures in seven dimensions

An SU(3)-structure on a 7D manifold M_7 is specified by a real one-form η , a real two-form J and a complex decomposable three-form Ω .³ The one-form η comes with a dual vector, X , such that $\iota_X \eta = 1$. These must be globally defined, and need to satisfy the algebraic constraints

$$\iota_X J = \iota_X \Omega = 0, \quad \Omega \wedge J = 0, \quad \Omega \wedge \bar{\Omega} \neq 0, \quad J \wedge J \wedge J \neq 0. \quad (2.1)$$

We will normalize Ω such that $\Omega \wedge \bar{\Omega} = -\frac{4i}{3} J \wedge J \wedge J$.

Since $\text{SU}(3) \subset \text{SO}(7)$, an SU(3) structure determines a metric on M_7 . This is constructed as follows. Locally, X specifies a 1D subspace, while η identifies a 6D, by definition orthogonal, subspace B_6 (by requiring its tangent space to be spanned by all vectors Y satisfying $\iota_Y \eta = 0$). From the first condition in (2.1), it follows that J and Ω live on B_6 . They will then induce a metric g_6 on B_6 just like a standard SU(3)-structure in 6D (see e.g. [55]). Denoting the line element of g_6 by $ds^2(B_6)$, the 7D line element can in the end be written as

$$ds^2(M_7) = ds^2(B_6) + \eta^2, \quad (2.2)$$

and the associated volume form is $\text{vol}_7 = \frac{1}{6} J \wedge J \wedge J \wedge \eta$.

SU(3)-structures are classified by their intrinsic torsion. This splits in torsion classes, which transform in irreducible representations of SU(3), and parameterize the exterior derivative of the invariant forms as [56, 57]

$$\begin{aligned} d\eta &= R J + T_1 + 2\text{Re}(\bar{V}_1 \lrcorner \Omega) + \eta \wedge W_0, \\ dJ &= \frac{3}{2} \text{Im}(\bar{W}_1 \Omega) + W_4 \wedge J + W_3 + \eta \wedge \left[\frac{2}{3} \text{Re} E J + T_2 + 2\text{Re}(\bar{V}_2 \lrcorner \Omega) \right], \\ d\Omega &= W_1 J \wedge J + W_2 \wedge J + \bar{W}_5 \wedge \Omega + \eta \wedge (E \Omega - 4V_2 \wedge J + S). \end{aligned} \quad (2.3)$$

Here, the real scalar R and the complex scalars W_1, E are SU(3)-singlets; T_1, T_2 are real two-forms, while W_2 is a complex two-form, all living on B_6 and transforming in the **8** of SU(3); W_3 is a real three-form, of type $(2, 1) + (1, 2)$ with respect to the almost complex structure I , and transforming in the **6** + $\bar{\mathbf{6}}$; S is a complex $(2, 1)$ -form in the **6**. Finally, V_1, V_2, W_5 are $(1, 0)$ -forms in the **3**, and W_0, W_4 are real one-forms in the **3** + $\bar{\mathbf{3}}$. The five torsion classes $W_k, k = 1, \dots, 5$, correspond to the better known torsion classes of 6D SU(3)-structures [58].

We remark that these torsion classes are not completely independent, as they must satisfy certain consistency constraints coming from the nilpotency of the exterior derivative

³A complex three-form is decomposable if it can be written as the wedge product of three complex one-forms.

(see e.g. (C.8)). It will also turn out that for the manifolds considered in this paper all the $SU(3)$ vector torsion classes, namely V_1, V_2, W_0, W_4 and W_5 , vanish.⁴

A given manifold M_7 can admit a whole space of $SU(3)$ -structures, which are distinguished by the different values of their torsion classes. When this is used as a compactification space of 11D supergravity, some specific configurations of the torsion classes are particularly relevant, in that they correspond to supersymmetry being preserved on an $AdS_4 \times M_7$ background. Indeed, the conditions for a supersymmetric AdS vacuum can be expressed as differential conditions on the structure forms, which in turn translate into restrictions on the torsion classes. As we prove in appendix A, when M_7 is homogeneous the only allowed maximally symmetric, supersymmetric solutions are of the Freund–Rubin type, which means the four-form flux of 11D supergravity is purely along AdS, and the metric on M_7 is Einstein. Within this class of solutions, for the manifolds of table 1 two classes of supersymmetric conditions arise, leading to vacua preserving a different fraction of supersymmetry.

The first class preserves $N = 2$ supersymmetry in 4D, and is characterized by non-vanishing singlet classes only, constrained in such a way that

$$d\eta = 2J, \quad dJ = 0, \quad d\Omega = 4i\eta \wedge \Omega. \quad (2.4)$$

This defines a Sasaki–Einstein structure on M_7 . Note that for a solution, X preserves η , J and the almost complex structure I . Hence it is a Killing vector of the metric (2.2). However, by evaluating $\mathcal{L}_X \Omega = \iota_X d\Omega = 4i\Omega$, we see that X rotates Ω by a phase; this corresponds to the R-symmetry of the $N = 2$ solution.

The other class leads to solutions that are invariant under the $U(1)$ generated by X , and which preserve $N = 1$ supersymmetry only. The $SU(3)$ torsion classes are constrained as

$$d\eta = \frac{i}{4}W_1J + iW_2, \quad dJ = \frac{3i}{2}W_1\text{Re}\Omega, \quad d\Omega = W_1J \wedge J + W_2 \wedge J, \quad (2.5)$$

with $\text{Re}W_1 = \text{Re}W_2 = 0$. A solution corresponds to a particular X -invariant weak G_2 holonomy, and can be described as a $U(1)$ -fibration (specified by η) over a 6D half-flat manifold whose $SU(3)$ -structure is described by J and Ω . The derivation of the conditions (2.4), (2.5) is given in appendix A.

In the last column of table 1, we specified which cosets allow which class of vacua. Interestingly, there are some cosets admitting vacua that belong to both classes. This happens because actually the manifold admits a further reduced structure (namely, a tri-Sasakian structure) preserving an enhanced amount of supersymmetry ($N = 3$, see section 5 for details).

⁴The appearance of such vector torsion classes on a homogeneous manifold would necessarily imply extra left-invariant vectors, and further reduce the $SU(3)$ -structure.

2.2 A finite-dimensional space of SU(3)-structures

In order to dimensionally reduce 11D supergravity on SU(3)-structure manifolds, we need to specify our truncation ansatz. Following previous studies in the literature (see e.g. [39, 40, 42]), we introduce a finite basis of differential forms in which the SU(3)-structure forms and the 11D supergravity three-form are to be expanded. Since the 7D metric follows from the SU(3)-structure, this will completely fix the truncation ansatz.

We assume that on M_7 there exists a set of real differential forms, made of: a single one-form θ , n_V two-forms ω_i , $2n_H$ three-forms α_A, β^A , n_V four-forms $\tilde{\omega}^i$, and a single six-form $\tilde{\omega}^0$, with the range of the indices being $i = 1, \dots, n_V$ and $A = 0, 1, \dots, n_H - 1$. The values of n_V and n_H are not fixed a priori in our general discussion, and will turn out to correspond to the number of vector multiplets and hypermultiplets in the 4D supergravity theory, respectively. Using these forms, we parameterize the family of SU(3)-structures on M_7 as

$$\eta = e^V \theta, \quad J = e^{-V} v^i \omega_i, \quad \Omega = e^{-\frac{3}{2}V} (Z^A \alpha_A - \mathcal{G}_A \beta^A), \quad (2.6)$$

where V, v^i are real and Z^A, \mathcal{G}_A are complex constant parameters.⁵ When in the next section we will discuss the dimensional reduction of 11D supergravity, these parameters will be promoted to scalar fields depending on the external space-time coordinates. Note that the expansion forms have no dependence on the parameters.⁶

We now list a set of sufficient conditions on the forms ensuring that (2.6) indeed defines a family of SU(3)-structures, and that the dimensional reduction of 11D supergravity on M_7 goes through, leading to $N = 2$ gauged supergravity in 4D. Similar relations have been discussed previously in the literature (see e.g. [39, 40, 42]), therefore our presentation will be quick. However, contrarily to e.g. [39, 42], our constraints will be given with no integration over M_7 . This strong assumption ensures in many cases that the reduction goes through at the level of the equations of motion as well, which implies consistency of the truncation. For the moment one should take our relations as assumptions on the geometry of a generic compact manifold M_7 . When in section 5 we will focus on coset manifolds G/H admitting SU(3)-structure, we will see that the assumptions we are making are all satisfied if one identifies the expansion forms precisely with the forms being invariant under the left-action of the group G . However, we stress that the consistent truncation we will establish is valid for some classes of non-homogeneous SU(3)-structures as well. For instance, a generic Sasaki–Einstein or tri-Sasakian structure also provides a set of forms with the correct properties. Likewise, the fibrations of Calabi–Yau three-folds over a circle studied in [43, 44] satisfy our constraints and thus lead to a consistent truncation, at least when just the universal volume modulus and no complex structure deformations are included in the truncation (which leads to $n_V = n_H = 1$).

⁵The factors of e^V in the expansions of J and Ω have been chosen for convenience, in order to ensure that v^i and Z^A become, respectively, the almost symplectic and almost complex structure moduli of the base B_6 . These naturally arise within a type IIA string frame setup.

⁶See [41] for a 6D analysis in which the basis forms are allowed to depend on the moduli.

The one-form θ comes with a dual vector k , such that $\iota_k \theta = 1$; hence we have the identification $X = e^{-V} k$. We assume that all the basis forms apart from θ live on the 6D space orthogonal to k ,

$$\iota_k \omega_i = \iota_k \alpha_A = \iota_k \beta^A = \iota_k \tilde{\omega}^i = 0. \quad (2.7)$$

Furthermore, we require

$$\omega_i \wedge \alpha_A = \omega_i \wedge \beta^A = \tilde{\omega}^i \wedge \alpha_A = \tilde{\omega}^i \wedge \beta^A = \alpha_A \wedge \alpha_B = \beta^A \wedge \beta^B = 0, \quad (2.8)$$

and

$$\omega_i \wedge \tilde{\omega}^j = -\delta_i^j \tilde{\omega}^0, \quad \alpha_A \wedge \beta^B = -\delta_A^B \tilde{\omega}^0. \quad (2.9)$$

These conditions on the basis forms imply that η , J and Ω constructed as in (2.6) obey the algebraic constraints (2.1). The 7D metric eq. (2.2) can be written as

$$ds^2(M_7) = e^{-V} ds^2(B_6) + e^{2V} \theta^2, \quad (2.10)$$

where $ds^2(B_6)$ is determined by J and Ω . Note that this 6D metric is actually rescaled with respect to the one given in (2.2). Eqs. (2.8), (2.9) also imply that there is a symplectic $\text{Sp}(2n_H, \mathbb{R})$ structure on the space of three-forms in our basis. The quantities in our dimensional reduction coming from an expansion in the three forms will inherit this symplectic structure. Collecting the three-forms in a vector Σ and defining the $\text{Sp}(2n_H, \mathbb{R})$ metric \mathbb{C} as

$$\Sigma = \begin{pmatrix} \beta^A \\ \alpha_A \end{pmatrix}, \quad \mathbb{C} = \begin{pmatrix} 0 & \delta_A^B \\ -\delta^A_B & 0 \end{pmatrix}, \quad (2.11)$$

we can adopt a convenient symplectic notation, in which the A, B indices are suppressed. Introducing furthermore the symplectic vector

$$Z = \begin{pmatrix} Z^A \\ \mathcal{G}_A \end{pmatrix}, \quad (2.12)$$

we can for instance write the expansion of Ω in (2.6) as $\Omega = e^{-\frac{3}{2}V} Z^T \mathbb{C} \Sigma$. This symplectic notation will be used throughout the paper.

In addition to the above requirements, we also need that in our basis the wedge product of a pair of two-forms can be expanded in the four-forms, with constant coefficients \mathcal{K}_{ijk} :

$$\omega_i \wedge \omega_j = \mathcal{K}_{ijk} \tilde{\omega}^k \quad \Rightarrow \quad \omega_i \wedge \omega_j \wedge \omega_k = -\mathcal{K}_{ijk} \tilde{\omega}^0. \quad (2.13)$$

Let us furthermore introduce the following useful contractions

$$\mathcal{K} = \frac{1}{6} \mathcal{K}_{ijk} v^i v^j v^k, \quad \mathcal{K}_i = \mathcal{K}_{ijk} v^j v^k, \quad (2.14)$$

in terms of which the 7D volume form can be written as

$$\text{vol}_7 = -e^{-2V} \mathcal{K} \tilde{\omega}^0 \wedge \theta. \quad (2.15)$$

Sometimes we need a canonical volume form independent of the choice of a particular metric and then we will take $-\tilde{\omega}^0 \wedge \theta$.

Moreover, we need that the set of basis three-forms be closed under the action of the 7D Hodge dual associated with the metric (2.10). While this is automatic for the even forms (see eq. (2.27) below), for the three-forms Σ we need to assume that the following relation holds:

$$*\Sigma = -\mathbb{M}\Sigma \wedge \eta, \quad (2.16)$$

where the matrix of coefficients \mathbb{M} is required not to depend on the coordinates on M_7 , but can depend on the metric parameters. Recalling $*^2 = id$, one can show that \mathbb{M} satisfies $\mathbb{M}^2 = -\mathbb{1}$ and that it is symplectic (i.e. $\mathbb{M}^T \mathbb{C} \mathbb{M} = \mathbb{C}$), hence it can be written as

$$\mathbb{M} = \begin{pmatrix} (\text{Im}\mathcal{M})^{-1} \text{Re}\mathcal{M} & -(\text{Im}\mathcal{M})^{-1} \\ \text{Im}\mathcal{M} + \text{Re}\mathcal{M}(\text{Im}\mathcal{M})^{-1} \text{Re}\mathcal{M} & -\text{Re}\mathcal{M}(\text{Im}\mathcal{M})^{-1} \end{pmatrix}, \quad (2.17)$$

where \mathcal{M}_{AB} is a complex, symmetric matrix.

Finally, we require that the set of basis forms is closed under exterior derivative:

$$\begin{aligned} d\theta &= m^i \omega_i, \\ d\omega_i &= q_i^j \theta \wedge \omega_j + Q_i^T \mathbb{C} \Sigma, & d\tilde{\omega}^i &= -\theta \wedge \tilde{\omega}^j q_j^i, & d\tilde{\omega}^0 &= 0, \\ d\Sigma &= \theta \wedge \mathbb{U} \Sigma - \tilde{\omega}^i Q_i, \end{aligned} \quad (2.18)$$

where we defined

$$Q_i = \begin{pmatrix} m_i^A \\ e_{iA} \end{pmatrix}, \quad \mathbb{U} = \begin{pmatrix} v^A{}_B & t^{AB} \\ s_{AB} & u_A{}^B \end{pmatrix}, \quad (2.19)$$

and $m^i, q_i^j, m_i^A, e_{iA}, u_A{}^B, s_{AB}, t^{AB}, v^A{}_B$ are real constants, with $s_{AB} = s_{BA}$, $t^{AB} = t^{BA}$, and $v^A{}_B = -u_B{}^A$. It follows that $\mathbb{U}^T \mathbb{C} + \mathbb{C} \mathbb{U} = 0$, i.e. \mathbb{U} is in the $\mathfrak{sp}(2n_H, \mathbb{R})$ algebra. Nilpotency of the exterior derivative imposes the constraints

$$\begin{aligned} m^i Q_i &= 0, & m^i q_i^j &= 0, & \mathbb{U} Q_i + q_i^j Q_j &= 0, \\ Q_i^T \mathbb{C} Q_j - q_{[i}^k \mathcal{K}_{j]kl} m^l &= 0, \end{aligned} \quad (2.20)$$

where the expression in the last line is obtained after noting that the exterior derivative of the expression on the right in (2.13), together with the fact that on a compact manifold the volume-form cannot be exact, yields

$$q_{(i}^l \mathcal{K}_{jk)l} = 0. \quad (2.21)$$

These constants are going to be the geometric fluxes in our dimensional reduction. While for the first part of the paper we will keep them generic, in the second part, when we will turn to our concrete examples, they will take specific (possibly zero) values.

The above relations completely characterize the $\text{SU}(3)$ -structure. The $\text{SU}(3)$ torsion classes can be evaluated in terms of the parameters $V, v^i, Z^A, \mathcal{G}_A$ and of the geometric fluxes. For instance, we find that the scalar torsion classes are given by

$$R = \frac{e^{2V} m^i \mathcal{K}_i}{6 \mathcal{K}}, \quad W_1 = \frac{\sqrt{2} e^{\frac{K_\Omega}{2} + \frac{V}{2}}}{3 \mathcal{K}^{1/2}} v^i Q_i^T \mathbb{C} Z, \quad E = -i e^{K_\Omega - V} Z^T \mathbb{C} \mathbb{U} \bar{Z}, \quad (2.22)$$

where K_Ω is given in eq. (2.23) below. The differential constraints on the basis forms also ensure that the $SU(3)$ vector torsion classes V_1, V_2, W_0, W_4, W_5 all vanish. More generally, the constraints ensure that no $SU(3)$ triplets appear in the reduction (see e.g. [40] for a discussion on the importance of this fact).

2.3 Special Kähler geometry on the space of deformations

In this section we will briefly discuss how the finite-dimensional space of $SU(3)$ -structures defined above is described by special Kähler geometry. More thorough discussions can be found e.g. in [40, 41]. For the purposes of this section, we can just focus on the forms J and Ω living on B_6 , since the scale of the one-form η turns out to decouple in the dimensional reduction and corresponds to a trivial extra deformation associated with the 4D dilaton. It is known that at each point of a 6D space, the deformations of the almost complex structure induced by a decomposable three-form Ω span a special Kähler space [59], and an analogous result holds for the deformations of the (complexified) almost symplectic form J . When the assumptions of the previous section are satisfied, these properties extend globally on the 6D (and 7D) manifold, in analogy with the well-known case of the moduli space of Calabi–Yau three-folds [60].

Let us first consider the almost complex structure deformations. Given an expansion of Ω in terms of a symplectic basis as in (2.6), the Z^A , $A = 0, 1, \dots, n_H$, are projective coordinates on the space of deformations, which has complex dimension n_H , while the \mathcal{G}_A are holomorphic functions of the Z^A . The \mathcal{G}_A are related to the prepotential function $\mathcal{G}(Z)$ as $\mathcal{G}_A = \partial\mathcal{G}/\partial Z^A$. Contrarily to the case of J -deformations (described below), constructing the prepotential requires some non-trivial extra work. In appendix B we solve this problem by showing how the special Kähler data can concretely be extracted given a symplectic basis of expansion forms. Namely, we provide a simple algorithm for obtaining the dependent variables \mathcal{G}_B as holomorphic functions of the independent variables Z^A , as well as the prepotential $\mathcal{G}(Z)$. Having this useful result at hand, the Kähler potential K_Ω is constructed as

$$e^{-K_\Omega} \tilde{\omega}^0 = -i e^{3V} \Omega \wedge \bar{\Omega} = i (\bar{Z}^A \mathcal{G}_A - Z^A \bar{\mathcal{G}}_A) \tilde{\omega}^0, \quad (2.23)$$

and the Kähler metric $g_{a\bar{b}}$ on the space of deformations is deduced by introducing (away from the $Z^0 = 0$ locus) special coordinates $z^a = Z^a/Z^0$, $a = 1, \dots, n_H$, and computing $g_{a\bar{b}} = \frac{\partial}{\partial z^a} \frac{\partial}{\partial \bar{z}^b} K_\Omega$.

The period matrix \mathcal{M}_{AB} , which relates the upper and lower components of the symplectic section as

$$\mathcal{G}_A = \mathcal{M}_{AB} Z^B, \quad D_c \mathcal{G}_A = \overline{\mathcal{M}}_{AB} D_c Z^B, \quad (2.24)$$

(where D_c is the Kähler covariant derivative) is computed in terms of the prepotential via the standard special Kähler geometry formula

$$\mathcal{M}_{AB} = \overline{\mathcal{G}}_{AB} + 2i \frac{(\text{Im}\mathcal{G})_{AC} Z^C (\text{Im}\mathcal{G})_{BD} Z^D}{(\text{Im}\mathcal{G})_{CD} Z^C Z^D}, \quad (2.25)$$

where $\mathcal{G}_{AB} = \partial^2 \mathcal{G} / \partial Z^A \partial Z^B$. One can prove that the matrix \mathcal{M} introduced in the expression (2.16) for the Hodge dual of the basis three-forms, is precisely identified with this period matrix (see e.g. [41]).

We now turn to the moduli space of the (complexified) almost symplectic form. Given the assumptions of the previous section, we show in the following that locally this is also a special Kähler manifold. The proof is analogous to the one for the Kähler structure moduli space of Calabi–Yau three-folds [61, 60], with the important difference that we do not need to integrate over the compact space due to the strong assumptions on the basis forms we made above.

A metric on the moduli space of the almost symplectic structure J (which, as we will see, arises naturally in the dimensional reduction) is defined by

$$g_{ij} = \frac{\omega_i \wedge * \omega_j}{4 e^{2V} \text{vol}_7}. \quad (2.26)$$

This can be computed using the identity

$$* \omega_i = -\eta \wedge J \wedge \omega_i + \frac{3}{2} \frac{\omega_i \wedge J \wedge J}{J \wedge J \wedge J} \eta \wedge J \wedge J, \quad (2.27)$$

which follows from the fact that we have an $\text{SU}(3)$ -structure and that the basis two-forms are of type (1,1) with respect to I (since (2.8) implies $\Omega \wedge \omega_i = 0$). Plugging in the expansion of J and using the properties of the basis forms, we obtain

$$g_{ij} = -\frac{1}{4\mathcal{K}} \mathcal{K}_{ijk} v^k + \frac{1}{16\mathcal{K}^2} \mathcal{K}_{ikl} v^k v^l \mathcal{K}_{jmn} v^m v^n. \quad (2.28)$$

We see that g_{ij} depends on the v^i metric parameters only, and therefore is indeed a metric on the space of deformations of J , expanded as in (2.6). In the dimensional reduction, the J -moduli v^i are naturally complexified with the real parameters b^i arising from expanding the 11D supergravity three-form A_3 as $A_3 = b^i \omega_i \wedge \theta + \dots$ (see eq. (3.3)), and one is led to introduce the complex combination

$$t^i = b^i + i v^i. \quad (2.29)$$

One immediately checks that (2.28) is a special Kähler metric with respect to these coordinates, namely $g_{ij} = \frac{\partial}{\partial t^i} \frac{\partial}{\partial t^j} K$, where the Kähler potential K is

$$e^{-K} = \frac{4}{3} \mathcal{K}_{ijk} v^i v^j v^k = 8\mathcal{K}. \quad (2.30)$$

This is the standard Kähler potential following from the cubic prepotential

$$\mathcal{F} = -\frac{1}{6} \mathcal{K}_{ijk} \frac{X^i X^j X^k}{X^0}, \quad (2.31)$$

through the special Kähler geometry formula $e^{-K} = i(\bar{X}^I \mathcal{F}_I - X^I \bar{\mathcal{F}}_I)$, once the projective coordinates X^I are identified as

$$X^I \equiv (X^0, X^i) = (X^0, X^0 t^i), \quad (2.32)$$

and with $\mathcal{F}_I = \partial\mathcal{F}(X)/\partial X^I$.⁷

In the dimensional reduction, the period matrix \mathcal{N}_{IJ} relating X^I and \mathcal{F}_J will also appear, playing the role of the gauge kinetic matrix. Starting from the prepotential (2.31) and implementing the formula (2.25), we obtain the explicit expressions

$$\begin{aligned} \text{Re } \mathcal{N}_{00} &= -\frac{1}{3}\mathcal{K}_{ijk}b^ib^jb^k, & \text{Re } \mathcal{N}_{0i} &= \frac{1}{2}\mathcal{K}_{ijk}b^jb^k, & \text{Re } \mathcal{N}_{ij} &= -\mathcal{K}_{ijk}b^k, \\ \text{Im } \mathcal{N}_{00} &= -\mathcal{K}(1 + 4g_{ij}b^ib^j), & \text{Im } \mathcal{N}_{0i} &= 4\mathcal{K}g_{ij}b^j, & \text{Im } \mathcal{N}_{ij} &= -4\mathcal{K}g_{ij}. \end{aligned} \quad (2.33)$$

3 The dimensional reduction

In this section, we present the reduction of the bosonic sector 11D supergravity on any 7D manifold admitting a basis of differential forms of the type described in section 2.2. In the following we provide in turn our truncation ansatz and the resulting 4D action, while the technical details of the reduction are given in appendix C. In section 4, we will prove that the reduced action agrees with the formalism of 4D $N = 2$ gauged supergravity coupled to n_V vector multiplets and n_H hypermultiplets. Then in section 5 we will apply our general results to the coset manifolds of table 1.

3.1 The truncation ansatz

The bosonic content of 11D supergravity [62] includes the metric and a three-form potential A_3 , with field strength $G_4 = dA_3$. The action is

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int \left(R * 1 - \frac{1}{2}G_4 \wedge *G_4 - \frac{1}{6}A_3 \wedge G_4 \wedge G_4 \right). \quad (3.1)$$

For the 11D metric we take the following truncation ansatz

$$ds^2 = e^{2V}\mathcal{K}^{-1}ds_4^2 + e^{-V}ds^2(B_6) + e^{2V}(\theta + A^0)^2. \quad (3.2)$$

where ds_4^2 is a metric on the 4D space-time, while the 7D part is like the one introduced in the previous section, with the parameters v^i, z^a, V now promoted to scalar fields on the 4D space-time. We are also gauging the reparametrizations of the coordinate associated with the vector dual to θ by a 4D one-form A^0 . Finally, we introduced a Weyl factor in front of the external metric in order to end up in 4D Einstein frame.

The ansatz for the three-form potential A_3 is defined by the most general expansion in the basis forms:

$$A_3 = C_3 + B \wedge (\theta + A^0) - A^i \wedge \omega_i + \xi^A \alpha_A - \tilde{\xi}_A \beta^A + b^i \omega_i \wedge (\theta + A^0), \quad (3.3)$$

⁷It is interesting to note that the deformations of J and Ω can be described on the same footing by using the formalism of generalized geometry. Indeed, after introducing the zero-form $\omega_0 = 1$ and the index $I = (0, i)$, the symplectic expansion of Ω in (2.6) finds a counterpart in the polyform $e^{t^i \omega_i} = X^I \omega_I - \mathcal{F}_I \tilde{\omega}^I$. Moreover, a symplectic structure on the space of even forms is defined by $\langle \omega_I, \tilde{\omega}^J \rangle = -\delta_I^J \tilde{\omega}^0$, where \langle, \rangle is the Mukai pairing, which in six dimensions is antisymmetric. It follows that the Kähler potential is given by $e^{-K} \tilde{\omega}^0 = -i \langle e^{t^i \omega_i}, e^{\bar{t}^i \omega_i} \rangle$, in analogy with (2.23). See e.g. [40] for details.

where C_3 is a three-form, B is a two-form, the A^i are one-forms, and $b^i, \xi^A, \tilde{\xi}_A$ are real scalars, all defined on the 4D space-time. Note that $\xi^A, \tilde{\xi}_A$ can be assembled in a symplectic vector

$$\xi = \begin{pmatrix} \xi^A \\ \tilde{\xi}_A \end{pmatrix}. \quad (3.4)$$

For the field strength G_4 we take

$$G_4 = dA_3 + G_4^{\text{flux}}, \quad (3.5)$$

with

$$G_4^{\text{flux}} = p^A \alpha_A \wedge \theta - q_A \beta^A \wedge \theta + e_i \tilde{\omega}^i, \quad (3.6)$$

where e_i, p^A, q_A are constants parameterizing the flux of the four-form on the internal manifold. Note that these are only non-trivial if they correspond to non-exact parts of G_4 ; otherwise, they can be reabsorbed through redefinitions of the fields $\xi, \tilde{\xi}, b$. It will be convenient to arrange the four-form fluxes p^A, q_A together with the geometric fluxes Q_i introduced in (2.18), (2.19) in a larger matrix Q_I , where we define a new index $I = (0, i) = 0, 1, \dots, n_V$. Namely, we introduce

$$Q_0 = \begin{pmatrix} p^A \\ q_A \end{pmatrix}, \quad \text{and} \quad Q_I = (Q_0, Q_i) = \begin{pmatrix} p^A & m_i^A \\ q_A & e_{iA} \end{pmatrix}. \quad (3.7)$$

The Bianchi identity $dG_4 = 0$ imposes the constraints

$$q_i^j e_j = 0, \quad Q_0^T \mathbb{C} Q_i = 0. \quad (3.8)$$

The three-form C_3 is non-dynamical in four dimensions; after having worked out the 4D action we will dualize it to a constant e_0 . We can then introduce

$$e_I = (e_0, e_i) \quad (3.9)$$

and, for convenience,

$$m^I = (0, m^i). \quad (3.10)$$

The dimensional reduction is then performed by plugging the above ansatz in the 11D action (3.1). The details on the reduction of the 11D Einstein–Hilbert term are given in section C.1 of the appendix, where we provide a formula for the reduction of the higher-dimensional Ricci scalar (eq. (C.3)) that also applies to more general setups than the one of interest in this paper. Note that for the reduction to work, we need to assume that the vector k dual to θ is Killing for some canonical metric. An important piece of the computation is the evaluation of the internal Ricci scalar R_7 in terms of the 4D scalars and of the geometric fluxes, which yields a very non-trivial contribution to the 4D scalar potential. In order to do so, we first found a formula for R_7 purely in terms of SU(3)-structure data; this is given in (C.9). Then we plugged in this formula the expansions defined in section 2.2, which lead in the end to the result displayed in eq. (C.23). The details on the reduction of the kinetic and Chern–Simons terms for the 11D three-form are given in section C.2 of the appendix. In particular, the expansion of the 11D field strength G_4 and the result of the dualization of C_3 are given in eqs. (C.24)-(C.26). Adding up all the contributions, we find that 11D supergravity reduces to the 4D action below.

3.2 The 4D action

The full 4D action is obtained by adding the contributions from the 11D Einstein–Hilbert term, eq. (C.21), and from the form sector, eq. (C.28). It reads

$$S = \frac{1}{\kappa_4^2} \int \left[\frac{1}{2} R_4 * 1 + \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{top}} - V * 1 \right], \quad (3.11)$$

where the 4D gravitational coupling constant is $\kappa_4^{-2} = \kappa_{11}^{-2} \int (-\tilde{\omega}^0 \wedge \theta)$, and the Hodge star uses the 4D metric ds_4^2 . The kinetic terms are

$$\begin{aligned} \mathcal{L}_{\text{kin}} = & \frac{1}{4} e^{-4\phi} dB \wedge *dB + \frac{1}{4} \text{Im} \mathcal{N}_{IJ} F^I \wedge *F^J + \frac{1}{4} \text{Re} \mathcal{N}_{IJ} F^I \wedge F^J \\ & + g_{ij} Dt^i \wedge *D\bar{t}^j + g_{a\bar{b}} Dz^a \wedge *D\bar{z}^{\bar{b}} + d\phi \wedge *d\phi - \frac{1}{4} e^{2\phi} (D\xi)^T \wedge *(\mathbb{C} \mathbb{M} D\xi), \end{aligned} \quad (3.12)$$

while the topological terms, coming from the 11D Chern–Simons term, are

$$\begin{aligned} \mathcal{L}_{\text{top}} = & -\frac{1}{4} dB \wedge \left[(2e_I + Q_I^T \mathbb{C} \xi) A^I - \xi^T \mathbb{C} D\xi \right] \\ & + \frac{1}{4} q_i^l \mathcal{K}_{jkl} A^i \wedge A^j \wedge (dA^k - m^k B) - \frac{1}{4} e_I m^I B \wedge B. \end{aligned} \quad (3.13)$$

Here, $A^I = (A^0, A^i)$ are the gauge vectors, with generalized field strength

$$F^I = DA^I - m^I B, \quad (3.14)$$

where

$$DA^0 = dA^0, \quad DA^i = dA^i - A^0 \wedge A^j q_j^i. \quad (3.15)$$

The gauge kinetic matrix \mathcal{N}_{IJ} was given in (2.33). Further, we recall that g_{ij} is the special Kähler metric (2.28) on the moduli space of the complexified almost symplectic structure parameterized by the t^i , while $g_{a\bar{b}}$ is the special Kähler metric on the space of almost complex structures parameterized by the z^a . Furthermore the symplectic matrix \mathbb{M} appearing in the last term of (3.12) was defined in (2.17).

The covariant derivatives of the scalars arising from the internal metric involve just the gauge field A^0 , and read

$$\begin{aligned} Dv^i &= dv^i - v^j q_j^i A^0, \\ DZ^A &= dZ^A + (-u_B^A Z^B + t^{AB} \mathcal{G}_B) A^0, \\ D\mathcal{G}_A &= d\mathcal{G}_A + (s_{AB} Z^B + u_A^B \mathcal{G}_B) A^0. \end{aligned} \quad (3.16)$$

Note that Dz^a immediately follows from DZ^A , recalling that we identify $z^a = Z^a/Z^0$.⁸

⁸ The transformation of (Z^A, \mathcal{G}_A) that A^0 is gauging as expressed above, is manifestly symplectic. Note that the transformation of the Z^A already implies a certain transformation of the \mathcal{G}_A , since the \mathcal{G}_A are just functions of the Z^A . A priori this does not take the same form as above, the consistency condition being

$$(s_{AB} Z^B + u_A^B \mathcal{G}_B) = \delta \mathcal{G}_A = \frac{\partial \mathcal{G}_A}{\partial Z^C} \delta Z^C = \mathcal{G}_{AC} (-u_B^C Z^B + t^{CB} \mathcal{G}_B). \quad (3.17)$$

However, in our case this will work out exactly, since, as we show in the appendix around (C.13), the transformation is a proper complex deformation.

The covariant derivatives of the scalars coming from the 11D three-form are

$$\begin{aligned} Db^i &= db^i - A^0 b^j q_j^i + A^j q_j^i, \\ D\xi &= d\xi + A^I Q_I + A^0 \mathbb{U}\xi. \end{aligned} \quad (3.18)$$

Recalling that we have $t^i = b^i + iv^i$, it follows that

$$Dt^i \equiv Db^i + iDv^i = dt^i - (t^j A^0 - A^j) q_j^i. \quad (3.19)$$

The real scalar ϕ , sometimes called the 4D dilaton, is defined as

$$e^{2\phi} = e^{3V} \mathcal{K}^{-1}, \quad (3.20)$$

and replaces in the 4D action the degree of freedom associated with the rescalings of η .

Finally, the scalar potential is $V = V_{\text{geo}} + V_{G_4}$, where V_{geo} is the contribution from the internal Ricci scalar, given in (C.23), and V_{G_4} is the contribution from the kinetic term of the M-theory three-form, given in (C.35). The expression we eventually find is:

$$\begin{aligned} V &= 8e^K g_{ij} v^k q_k^i v^l q_l^j - 2e^{K+2\phi} (X^I Q_I^T + \xi^T \mathbb{U}^T) \mathbb{C}\mathbb{M} (Q_J \bar{X}^J + \mathbb{U}\xi) \\ &\quad + 4e^{K+K_\Omega+2\phi} (g^{ij} - 4v^i v^j) (Q_i^T \mathbb{C}Z) (Q_j^T \mathbb{C}\bar{Z}) \\ &\quad + 8ie^{K+K_\Omega} Z^T \mathbb{U}^T \mathbb{C}\mathbb{U}\bar{Z} + 8e^{K+2K_\Omega} (Z^T \mathbb{C}\mathbb{U}\bar{Z})^2 + 4e^{K+K_\Omega+2\phi} (m^i \mathcal{K}_i) (Z^T \mathbb{C}\mathbb{U}\bar{Z}) \\ &\quad - \frac{1}{4} e^{4\phi} [m^I \text{Im}\mathcal{N}_{IJ} m^J + (\mathcal{E}_I - \text{Re}\mathcal{N}_{IK} m^K) (\text{Im}\mathcal{N})^{-1IJ} (\mathcal{E}_J - \text{Re}\mathcal{N}_{JL} m^L)], \end{aligned} \quad (3.21)$$

where we defined

$$\mathcal{E}_I = e_I + Q_I^T \mathbb{C}\xi - \frac{1}{2} \delta_I^0 \xi^T \mathbb{C}\mathbb{U}\xi. \quad (3.22)$$

Let us remark a few interesting features of the action above, also comparing with previous work in the literature. Whenever the charges q_i^j are non-vanishing, we see that the vector covariant derivatives (3.15) have a non-abelian structure. Note that the q_i^j charges arise in (2.18) from a non-trivial dependence of the forms $\omega_i, \tilde{\omega}^i$ on the coordinate on M_7 parameterizing the direction of the vector X . It follows that this non-abelian structure is a specific feature of M-theory on 7D manifolds with $\text{SU}(3)$ -structure. This was first found in [43], and further studied in [44], for M-theory compactifications on Calabi–Yau three-folds fibered over a circle. Ref. [44] also studied the couplings of the scalars associated with the hypermultiplet sector (which was not analysed in [43]), including the charges appearing in our matrix \mathbb{U} . Here, we are extending these results to more general internal manifolds, and our 4D action reproduces the ones in [43, 44] once we take $Q_I = e_I = m^I = 0$ and the two-form B is dualized to a scalar. The full gauge algebra of our model will be discussed in section 4.2, while in section 5 we will see that one of our coset consistent truncations (the one on $N(1, -1)$) exhibits the non-abelian structure discussed here.

We can also make contact with similar reductions of type IIA supergravity on 6D manifolds with $\text{SU}(3)$ -structure, leading to $N = 2$ supergravity [39, 63, 64, 40, 6, 65]. In order to do this, we consider a reduction along the vector k dual to θ . This is allowed if k

preserves our truncation ansatz, or at least a consistent subtruncation of it. Actually, since we want off-shell supersymmetry to be preserved, not only the metric, but also our $SU(3)$ -structure forms η , J and Ω need to be preserved, i.e. we have to impose $\mathcal{L}_k \eta = \mathcal{L}_k J = \mathcal{L}_k \Omega = 0$. Let us consider e.g. J : vanishing of the Lie derivative leads to $v^{\hat{i}} q_{\hat{i}}^{\hat{j}} = 0$, where we are appending a hat to the indices surviving the subtruncation; since we want the $v^{\hat{i}}$ to be independent, we conclude that the charges $q_{\hat{i}}^{\hat{j}}$ need to vanish. Similar reasoning leads to require that the \mathbb{U} charges associated with the fields in the subtruncation have to vanish. It follows that in the type IIA reduction ansatz the v , b , z scalars become uncharged, and their covariant derivative becomes trivial. As for the remaining charges, one has that p^A and q_A describe the type IIA NSNS three-form flux, while the e_I correspond to the RR four-form flux, and m^i is the RR two-form flux. This reproduces the action arising from dimensional reduction of type IIA on 6D $SU(3)$ -structure manifolds, in the limit of vanishing Romans mass, $m^0 = 0$. In this context, switching on the Romans mass is trivially achieved by promoting $m^I = (0, m^i)$ to (m^0, m^i) . We will speculate about turning on m^0 in more general frameworks in section 6. We conclude that with respect to type IIA on 6D $SU(3)$ -structures, the present M-theory setup allows for extra couplings, which lead to non-abelian gauge groups and to a novel scalar potential including more terms than the ones previously appeared in the literature.

As a final remark, note that the whole action is invariant under $Sp(2n_H, \mathbb{R})$. Invariance under the electric-magnetic group $Sp(2n_V + 2, \mathbb{R})$ is present just for some terms, like the last line of (3.21) (once the symplectic vector $(m^I, \mathcal{E}_I)^T$ is defined).

4 The $N = 2$ gauging

4.1 The action with magnetic vectors

We are now going to show that the 4D action above is consistent with the bosonic sector of $N = 2$ gauged supergravity. A delicate point in this proof is how to treat the two-form B , since two-forms do not appear in standard matter-coupled $N = 2$ supergravity with purely electric gauging (see [66] for a review). This issue is not present when the parameters m^I all vanish (i.e. the one-form θ is closed), since in this case B enters in the action only through its field-strength dB , and therefore can easily be dualized to a scalar. After this transformation, the action is straightforwardly shown to be an electrically gauged $N = 2$ supergravity coupled to n_V vector multiplets and n_H hypermultiplets. However, in all the concrete examples we will study in the next section, the m^I are non-vanishing, implying that the potential B appears explicitly in the action; this makes it more involved to dualize it, a preliminary electric-magnetic symplectic transformation on the vector fields being required. In order to avoid these complications, which would also affect the gauge kinetic matrix and the prepotential \mathcal{F} of our model, we chose to remain in the same electric-magnetic frame naturally defined by the dimensional reduction, and to follow the approach of [67] (see also [68]). In this approach, the m^I are seen as charges under magnetic vectors

\tilde{A}_I , and there is no need to completely dualize the B -field away, since both the two-forms and their dual scalars play a role in the action. This is allowed because while the electric vectors and the scalars are dynamical fields with second-order equations of motion, both the magnetic vectors and the two-forms are treated as non-propagating, auxiliary fields. The role of the \tilde{A}_I is to make the magnetic gauging manifest, while the two-forms are required in order to ensure closure of the gauge algebra.⁹

In order to match the formalism of [67], we need to cast the action obtained from dimensional reduction in an equivalent form, in which the two-form B becomes non-dynamical, and its dual scalar, a , is introduced. This is achieved by the new action

$$\tilde{S} = \frac{1}{\kappa_4^2} \int \left[\frac{1}{2} R_4 * 1 + \tilde{\mathcal{L}}_{\text{kin}} + \tilde{\mathcal{L}}_{\text{top}} - V * 1 \right], \quad (4.1)$$

in which \mathcal{L}_{kin} and \mathcal{L}_{top} of section 3.2 are replaced by

$$\begin{aligned} \tilde{\mathcal{L}}_{\text{kin}} = & \frac{1}{4} \text{Im} \mathcal{N}_{IJ} F^I \wedge * F^J + \frac{1}{4} \text{Re} \mathcal{N}_{IJ} F^I \wedge F^J + g_{ij} D t^i \wedge * D \bar{t}^j \\ & + g_{a\bar{b}} D z^a \wedge * D \bar{z}^{\bar{b}} + d\phi \wedge * d\phi - \frac{1}{4} e^{2\phi} (D\xi)^T \wedge * (\mathbb{C} \mathbb{M} D\xi), \\ & + \frac{1}{4} e^{4\phi} \left(Da + \frac{1}{2} \xi^T \mathbb{C} D\xi \right) \wedge * \left(Da + \frac{1}{2} \xi^T \mathbb{C} D\xi \right) \end{aligned} \quad (4.2)$$

and by

$$\tilde{\mathcal{L}}_{\text{top}} = \frac{1}{4} q_i^I \mathcal{K}_{jkl} A^i \wedge A^j \wedge (dA^k - m^k B) - \frac{1}{4} e_I m^I B \wedge B - \frac{1}{2} B \wedge m^I d\tilde{A}_I, \quad (4.3)$$

where we defined

$$Da = da - A^I (e_I + \frac{1}{2} Q_I^T \mathbb{C} \xi) - \tilde{A}_I m^I, \quad (4.4)$$

\tilde{A}_I being the magnetic vector fields. Note that the only changes in $\tilde{\mathcal{L}}_{\text{kin}}$ and $\tilde{\mathcal{L}}_{\text{top}}$ with respect to \mathcal{L}_{kin} and \mathcal{L}_{top} are that the dB terms have been substituted by the kinetic term for a and a topological coupling between B and the magnetic vectors, corresponding to the last terms of (4.2) and of (4.3), respectively. The electric field strengths F^I and the covariant derivatives of the other scalars displayed in the previous section remain the same, and the scalar potential (3.21) is also unmodified. The relation between the new and the old action is seen by integrating out the vector field $m^I \tilde{A}_I$. Indeed, its equation of motion reads

$$Da + \frac{1}{2} \xi^T \mathbb{C} D\xi = -e^{-4\phi} * dB, \quad (4.5)$$

which states the duality relation between B and a . Recalling (4.4), we can solve for $m^I \tilde{A}_I$. Substituting the solution in (4.2), (4.3), the original action of section 3.2 is retrieved. Note

⁹The appearance of magnetic charges and of two-forms in flux compactifications was first discussed in a type II supergravity context, in [69] and in [70], respectively. An alternative way to match gauged $N = 2$ supergravity would be to invoke the formalism of [71, 72], where the coupling with tensor multiplets was constructed. In this approach, the two-forms are dynamical, the magnetic charges enter in their mass and topological terms, and their dual scalars do not appear in the action. These are precisely the features of the action we obtained from the dimensional reduction. The reason why we preferred the approach described above is that it makes the magnetic gauging manifest.

that, as promised, in the new action the two-form B is non-propagating, in that its kinetic term has been replaced by the kinetic term for the scalar a . The equation of motion of B is thus of first-order, and gives the duality relation between the electric vectors A^I and the magnetic vectors \tilde{A}_I , contracted with m^I .

We are now in the position of discussing consistency with $N = 2$ supergravity in the presence of both electric and magnetic gaugings. When the charges all vanish, our action is in straightforward agreement with the bosonic sector of ungauged $N = 2$ supergravity, with the gravity multiplet — comprising the 4D metric and the graviphoton A^0 — coupled to n_V vector multiplets $\{A^i, t^i\}$ and n_H hypermultiplets defined by the scalar fields $q^u = \{\phi, a, z^a, \xi^A, \tilde{\xi}_A\}$, with $u = 1, \dots, 4n_H$. From (4.2), we see that the kinetic terms of the hyperscalars define a σ -model with target space metric

$$h_{uv} dq^u dq^v = d\phi^2 + g_{a\bar{b}} dz^a d\bar{z}^{\bar{b}} + \frac{1}{4} e^{4\phi} \left(da + \frac{1}{2} \xi^T \mathbb{C} d\xi \right)^2 - \frac{1}{4} e^{2\phi} d\xi^T \mathbb{C} m d\xi, \quad (4.6)$$

which, as familiar from Calabi–Yau compactifications, is the quaternionic-Kähler metric obtained from the special Kähler manifold spanned by the z^a via the c-map construction of [73].¹⁰

Let us now turn to non-vanishing charges.

4.2 The gauging

The general scalar covariant derivatives in $N = 2$ gauged supergravity read

$$\begin{aligned} Dt^j &= dt^j + k_I^j A^I, \\ Dq^u &= dq^u + k_I^u A^I - \tilde{k}^{Iu} \tilde{A}_I. \end{aligned} \quad (4.7)$$

Here, we are allowing the isometries in the hyperscalar manifold to be gauged both electrically and magnetically by Killing vectors k_I^u and \tilde{k}^{Iu} , respectively, while for our purposes it will be enough to assume that the isometries in the vector multiplet scalar manifold are gauged just by electric vectors k_I^j .

From (3.19) we read off that the gauging of the vector multiplet scalars is described by the holomorphic Killing vectors

$$k_0^i = -t^j q_j^i, \quad k_j^i = q_j^i. \quad (4.8)$$

One can check that these vectors are Killing by using the constraint (2.21). The associated Killing prepotentials, satisfying [66]

$$i\mathcal{P}_I = k_I^j \partial_j K = -\bar{k}_I^{\bar{j}} \partial_{\bar{j}} K, \quad (4.9)$$

¹⁰From comparing our gauge kinetic terms with the ones in [66], we see that the normalization of our vectors A^I differs by a factor of $\sqrt{2}$ from the one in [66], namely $A^{\text{here}} = \sqrt{2} A^{\text{there}}$. From eq. (4.7) below we see that this affects the Killing vectors gauging the isometries of the scalar manifold: $k^{\text{here}} = k^{\text{there}} / \sqrt{2}$. It follows that the general formula for the $N = 2$ scalar potential, which is quadratic in these Killing vectors, acquires an extra factor of 2 with respect to the standard one, see eq. (D.1).

are

$$\mathcal{P}_0 = -\frac{b^j q_j^i \mathcal{K}_i}{4\mathcal{K}}, \quad \mathcal{P}_i = \frac{q_i^j \mathcal{K}_j}{4\mathcal{K}}, \quad (4.10)$$

where to compute \mathcal{P}_0 we used $v^i q_j^i \mathcal{K}_i = 0$, which follows from (2.21).

We now turn to the gauging of isometries in the hyperscalar manifold. From (3.16) we read off that the geometric u, t and s -charges in the \mathbb{U} matrix lead to gauging under the graviphoton A^0 of a Killing vector with action

$$\delta Z = \mathbb{U}Z, \quad \delta \xi = \mathbb{U}\xi. \quad (4.11)$$

The vector can be written as

$$k_{\mathbb{U}} = (\mathbb{U}Z)^A \frac{\partial}{\partial Z^A} + (\mathbb{U}\bar{Z})^A \frac{\partial}{\partial \bar{Z}^A} + (\mathbb{U}\xi)^A \frac{\partial}{\partial \xi^A} + (\mathbb{U}\xi)_A \frac{\partial}{\partial \tilde{\xi}_A}, \quad (4.12)$$

where for the special Kähler base of the quaternionic manifold we are using the projective coordinates Z^A rather than the special coordinates $z^a = Z^a/Z^0$. Such a transformation leaves the Kähler potential K_{Ω} invariant and leads thus indeed to a Killing vector of the special Kähler base. By also working out the transformation of the matrix \mathbb{M} under (4.11), one can check that our vector is Killing for the full quaternionic metric (4.6) (see [74, 44] for more details).

By looking again at the scalar covariant derivatives, we infer that the remaining flux charges m^I, e_I, Q_I are associated with a gauging of the Heisenberg algebra of isometries which exists for any quaternionic metric of the form (4.6). This algebra is generated by the Killing vectors

$$h^A = \frac{\partial}{\partial \tilde{\xi}_A} + \frac{1}{2} \xi^A \frac{\partial}{\partial a}, \quad h_A = \frac{\partial}{\partial \xi^A} - \frac{1}{2} \tilde{\xi}_A \frac{\partial}{\partial a}, \quad h = \frac{\partial}{\partial a}, \quad (4.13)$$

satisfying the Heisenberg commutation relations $[h_A, h^B] = \delta_A^B h$. Also taking into account (4.12), we find that the complete Killing vectors k_I^u and \tilde{k}^{Iu} , respectively gauging the quaternionic isometries electrically and magnetically, read

$$\begin{aligned} k_I^u \frac{\partial}{\partial q^u} &= \delta_I^0 k_{\mathbb{U}} + Q_I^A h_A + Q_{IA} h^A - e_I h, \\ \tilde{k}^{Iu} \frac{\partial}{\partial q^u} &= -m^I h. \end{aligned} \quad (4.14)$$

We now compute the Killing prepotentials $\mathcal{P}_I^x, \tilde{\mathcal{P}}^{xI}$, $x = 1, 2, 3$, associated with the quaternionic isometries being gauged. These are important data of $N = 2$ supergravity, in that they appear both in the formula for the $N = 2$ scalar potential and in the supersymmetry variations of the fermions. In order to perform the computation, we need the connection ω^x on the $SU(2)$ -bundle of the quaternionic-Kähler manifold with metric (4.6). We find that this is given by [73]

$$\begin{aligned} \omega^1 + i\omega^2 &= \sqrt{2} e^{\frac{K_{\Omega}}{2} + \phi} Z^T \mathbb{C} d\xi, \\ \omega^3 &= \frac{e^{2\phi}}{2} \left(da + \frac{1}{2} \xi^T \mathbb{C} d\xi \right) - 2e^{K_{\Omega}} \text{Im} \left(Z^A \text{Im} \mathcal{G}_{AB} d\bar{Z}^B \right). \end{aligned} \quad (4.15)$$

One can verify that our Killing vectors preserve this connection, and hence the SU(2)-curvature $\Omega^x = d\omega^x + \frac{1}{2}\epsilon^{xyz}\omega^y \wedge \omega^z$:

$$\mathcal{L}_{k_I}\omega^x = 0 \quad \Rightarrow \quad \mathcal{L}_{k_I}\Omega^x = 0 \quad (4.16)$$

(same for \tilde{k}^I). Due to this property, the general formula for the Killing prepotentials takes the particularly simple form:

$$\mathcal{P}_I^x = k_I^u \omega_u^x, \quad \tilde{\mathcal{P}}^{xI} = \tilde{k}^{uI} \omega_u^x. \quad (4.17)$$

This yields the following non-vanishing prepotentials:

$$\begin{aligned} \mathcal{P}_I^1 + i\mathcal{P}_I^2 &= \sqrt{2} e^{\frac{K\Omega}{2} + \phi} Z^T \mathbb{C} (Q_I + \delta_I^0 \mathbb{U}\xi), \\ \mathcal{P}_I^3 &= -\frac{1}{2} e^{2\phi} (e_I + Q_I^T \mathbb{C}\xi - \frac{1}{2} \delta_I^0 \xi^T \mathbb{C}\mathbb{U}\xi) - \delta_I^0 e^{K\Omega} Z^T \mathbb{C}\mathbb{U}\bar{Z}, \\ \tilde{\mathcal{P}}^{3I} &= -\frac{1}{2} e^{2\phi} m^I. \end{aligned} \quad (4.18)$$

We have thus specified the isometries being gauged in the scalar covariant derivatives appearing in our action (4.1). These unify gaugings previously discussed in the literature in different contexts: the gaugings in the vector multiplet sector generated by the Killing vectors (4.8) were discovered in [43], and further discussed in [44], for M-theory compactifications on Calabi–Yau three-folds fibered over a circle; the gauging of the isometry (4.11) was also described in [44]. While the gauging of the Heisenberg algebra (4.13) was not discussed in [43, 44], it is familiar from flux compactifications of type II supergravity on 6D manifolds with SU(3) or SU(3)×SU(3) structure, see e.g. [63, 40, 75, 76, 77]. Above we have shown that for M-theory on generic SU(3)-structure manifolds, all these gaugings can possibly be switched on, provided the constraints (2.20), (2.21) and (3.8) are satisfied. This leads to non-abelian gauge groups and new scalar potentials.

The full Killing vectors gauging both the vector- and hypermultiplet sector are obtained by adding up the Killing vectors in eq. (4.8) and in eq. (4.14):

$$k_I = k_I^j \frac{\partial}{\partial t^j} + \bar{k}_I^{\bar{j}} \frac{\partial}{\partial \bar{t}^{\bar{j}}} + k_I^u \frac{\partial}{\partial q^u}, \quad \tilde{k}^I = \tilde{k}^{uI} \frac{\partial}{\partial q^u}. \quad (4.19)$$

from these vectors we can infer the gauge algebra of our model. We already saw in section 3.2 that when the charges q_i^j are non-vanishing, the electric gauge vectors have non-abelian covariant derivatives (3.15). This non-abelian structure can be further specified by evaluating the commutators of our gauged isometries. Recalling the constraints (2.20), (2.21) and (3.8), we find that the non-vanishing commutators are:

$$\begin{aligned} [k_0, k_i] &= q_i^j k_j, \\ [k_i, k_j] &= -q_{[i}^k \mathcal{K}_{j]kl} \tilde{k}^l. \end{aligned} \quad (4.20)$$

As for the two-form B , it enters in the generalized field strengths F^I and in the topological term (4.3) precisely as prescribed by [67]. For vanishing m^I , a topological term

like the one in (4.3) was first found in $N = 2$ supergravity in [78], and in our context it was also discussed in [43, 44]. Its role is to ensure gauge invariance by compensating the transformation of the gauge kinetic matrix $\text{Re}\mathcal{N}$.

It remains to discuss the scalar potential. In $N = 2$ supergravity, this is completely determined by the gauging. In appendix D, we prove that the expression for the scalar potential following from the gauging described above precisely matches the one in (3.21) obtained from the dimensional reduction. We can therefore conclude that our general dimensional reduction is fully compatible with gauged $N = 2$ supergravity.

5 Explicit models from coset reduction

Up to now we have kept the discussion general. We have shown that $D = 11$ supergravity on any 7D manifold M_7 , not necessarily homogeneous, equipped with the particular $\text{SU}(3)$ -structure described in section 2, reduces to the general 4D $N = 2$ supergravity described in sections 3 and 4. It is somewhat tedious, but otherwise straightforward, to verify that the set (or a subset thereof) of the LI forms on all the cosets of table 1 fulfill all the requirements described in section 2. We have provided some details in appendix E. For an overview of all the models see figure 1. To determine the particular effective theory corresponding to each coset one only needs to specify the corresponding scalar manifolds, and the scalar charges imposed by the geometric and background fluxes. We analyse both issues in subsections 5.1 and 5.2, where we will also explicitly write out the scalar potential for some of the models. Further comments on the individual models are made in subsection 5.3 (where, in particular, we find a new, non-supersymmetric solution). Finally, in subsection 5.4 we describe the supersymmetric critical structure of our coset effective theories.

5.1 The scalar manifolds

Let us first discuss the vector multiplet and hypermultiplet scalar manifolds corresponding to the individual coset models, as given in table 2. As follows from our results of section 2 and appendix E, the LI moduli spaces of Kähler and complex structure deformations are, respectively, special Kähler and special quaternionic. For our cosets, the respective complex and quaternionic dimensions, n_V and n_H , are given by the number of LI two-forms and half the number of LI three-forms, respectively. These numbers are obtained on a case-by-case basis in appendix E, and we have summarised them in table 2. Let us now further show that the corresponding spaces are the homogeneous spaces quoted in that table.

The vector multiplet scalars are governed by the cubic prepotential (2.31). Therefore, once the constant, symmetric tensor \mathcal{K}_{ijk} has been specified for each model, the resulting special Kähler space in table 2 follows from the classification of [79]. To see this more explicitly, focus first on the cosets with $n_V = 1, 2$, or 3 , that is, all except $N(1, -1)$.

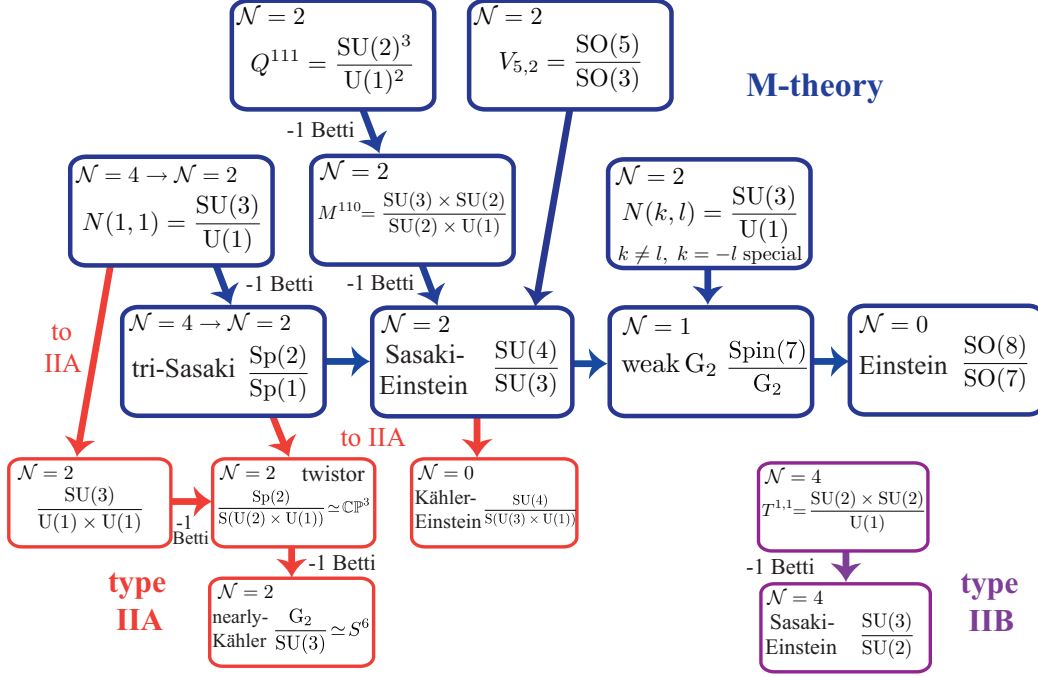


Figure 1: Web of interrelated consistent truncations of 11D (blue), type IIA (red) and type IIB (purple) supergravity. The diagram is based on figure 1 in [13], but updated with the coset models discussed in this paper. In each box we display the geometric structure (if any) and the coset manifold on which the truncation is based, as well as its amount of supersymmetry. Each arrow denotes a consistent subtruncation. Note that the Nearly-Kähler truncation cannot be reached directly from the $N(1,1)$ and $Sp(2)/Sp(1)$ models, since the respective choices of m^i geometric fluxes are incompatible.

From the explicit parametrization of appendix E, where $v^i = e^{2u_i}$, it follows that the complexified Kähler moduli t^i in (2.29) do take values on n_V copies of the upper half plane, $SU(1,1)/U(1)$. With the specific tensors \mathcal{K}_{ijk} quoted in appendix E, it is then easy to evaluate the Kähler metric (2.28). For $n_V = 3$, the corresponding line element is

$$ds^2 = \sum_{i=1}^3 \left[(du_i)^2 + \frac{1}{4} e^{-2u_i} (db^i)^2 \right]. \quad (5.1)$$

The cases $n_V = 2$ and $n_V = 1$ are recovered from (5.1) by setting $t^3 = t^1$, and $t^3 = t^1 = t^2$, respectively. The only remaining case we need to consider is $N(1, -1)$, for which $n_V = 5$. For this case, from the tensor \mathcal{K}_{ijk} in (E.73), and the generic expression (2.31) for the prepotential, we obtain that quoted in table 2. The corresponding Kähler potential is

$$K = -\log[i (\bar{t}^3 - t^3) ((t^1 - \bar{t}^1)(t^2 - \bar{t}^2) - (t^4 - \bar{t}^4)^2 - (t^5 - \bar{t}^5)^2)], \quad (5.2)$$

which is well-known (see e.g. [43]) to give rise to the special Kähler manifold given in the table. The corresponding Kähler metric, which we omit, can be straightforwardly

recovered from (5.2). A real parametrization for this metric is provided in terms of (E.75). The rest of the vector multiplet quantities for all our models, namely, the gauge kinetic and topological terms, can be recovered for each coset model by inserting these metrics and the tensors \mathcal{K}_{ijk} quoted in the appendices into the general expressions (2.33).

Let us now turn to the hypermultiplet sector. We found in section 4 that, after dualisation, the hyperscalars are described by the metric (4.6) on a special quaternionic-Kähler manifold. Recall that these are quaternionic-Kähler manifolds in the image of the c-map, and are completely determined by their special Kähler submanifold. The latter is empty for $n_H = 1$, as corresponds to reductions with rigid complex structure, and leads to the universal hypermultiplet moduli space, $SU(2,1)/S(U(2) \times U(1))$. For the cosets with $n_H = 2$, the complex structure does admit LI fluctuations, and we find two possibilities. Both have special Kähler submanifold $SU(1,1)/U(1)$, but equipped with either a cubic or a quadratic prepotential, which leads respectively to $G_{2(2)}/SO(4)$ or $SO(4,2)/(SO(4) \times SO(2))$ [73]. From the explicit calculation of the prepotentials in appendices B and E, we conclude that the $V_{5,2}$ model leads to the former case, while the $Sp(2)/Sp(1)$ and $N(1,1)$ models lead to the second. We have quoted the prepotentials themselves in table 2, written out in the symplectic frames introduced in appendices E.4, for $V_{5,2}$, and E.5, for $Sp(2)/Sp(1)$ and $N(1,1)$.

We can now explicitly find the metrics on all three special quaternionic manifolds, that define the hypermultiplet non-linear sigma model after dualisation of the two-form B , by evaluating the general metric (4.6). For $SU(2,1)/S(U(2) \times U(1))$, $z^a = 0$ and $\mathbb{M} = \mathbb{C}$, so (4.6) reduces to

$$h_{uv}dq^u dq^v = (d\phi)^2 + \frac{1}{4}e^{4\phi} \left[da + \frac{1}{2}(\xi d\tilde{\xi} - \tilde{\xi} d\xi) \right]^2 + \frac{1}{4}e^{2\phi}(d\xi)^2 + \frac{1}{4}e^{2\phi}(d\tilde{\xi})^2. \quad (5.3)$$

For the $n_H = 2$ cases, we parametrize the (only) complex coordinate z on the special Kähler submanifold $SU(1,1)/U(1)$ in terms of real scalars χ, φ as in (E.45), for $V_{5,2}$, and (E.61), for $Sp(2)/Sp(1)$ and $N(1,1)$. Then we compute the matrix \mathbb{M} entering the metric (4.6) in the specific symplectic frames introduced in appendices E.4, for $V_{5,2}$, and E.5, for $Sp(2)/Sp(1)$ and $N(1,1)$. In order to do so, we first compute the matrix \mathcal{M} in (2.25) with the help of the cubic, (E.43), and quadratic, (E.59), prepotentials, and using the holomorphic sections Z^A, G_A that can be read off from the expressions for Ω in (E.44) and (E.60). After a final rearrangement, the end result is, for the metric on $G_{2(2)}/SO(4)$,

$$\begin{aligned} h_{uv}dq^u dq^v = & \\ & d\phi^2 + 3(d\varphi)^2 + \frac{3}{4}e^{4\varphi}(d\chi)^2 + \frac{1}{4}e^{4\phi} \left[da + \frac{1}{2}(\xi^0 d\tilde{\xi}_0 + \xi^1 d\tilde{\xi}_1 - \tilde{\xi}_0 d\xi^0 - \tilde{\xi}_1 d\xi^1) \right]^2 \\ & + \frac{1}{4}e^{2\phi-6\varphi}(d\xi^0)^2 + \frac{1}{4}e^{2\phi-2\varphi} \left[d\xi^1 - \sqrt{3}\chi d\xi^0 \right]^2 \\ & + \frac{1}{4}e^{2\phi+2\varphi} \left[d\tilde{\xi}_1 - \sqrt{3}\chi^2 d\xi^0 + 2\chi d\xi^1 \right]^2 \\ & + \frac{1}{4}e^{2\phi+6\varphi} \left[d\tilde{\xi}_0 + \sqrt{3}\chi d\tilde{\xi}_1 - \chi^3 d\xi^0 + \sqrt{3}\chi^2 d\xi^1 \right]^2 \end{aligned} \quad (5.4)$$

and for the metric on $SO(4, 2)/(SO(4) \times SO(2))$,

$$\begin{aligned}
h_{uv} dq^u dq^v = & \\
& d\phi^2 + \frac{1}{4} d\varphi^2 + \frac{1}{4} e^{-2\varphi} (d\chi)^2 + \frac{1}{4} e^{4\phi} \left[da + \frac{1}{2} (\xi^0 d\tilde{\xi}_0 + \xi^1 d\tilde{\xi}_1 - \tilde{\xi}_0 d\xi^0 - \tilde{\xi}_1 d\xi^1) \right]^2 \\
& + \frac{1}{8} e^{2\phi+\varphi} (d\xi^0 + d\xi^1)^2 + \frac{1}{8} e^{2\phi+\varphi} (d\tilde{\xi}_0 - d\tilde{\xi}_1)^2 \\
& + \frac{1}{8} e^{2\phi-\varphi} \left[d\xi^0 - d\xi^1 + \chi (d\tilde{\xi}_0 - d\tilde{\xi}_1) \right]^2 \\
& + \frac{1}{8} e^{2\phi-\varphi} \left[d\tilde{\xi}_0 + d\tilde{\xi}_1 - \chi (d\xi^0 + d\xi^1) \right]^2.
\end{aligned} \tag{5.5}$$

We have verified that these three metrics are Einstein, as corresponds to quaternionic-Kähler spaces, with normalisation such that the Ricci tensor equals $-2(n_H + 2)$ times the metric. We have also doublechecked that these are the (only, once the normalisation has been fixed) G -invariant metrics on the corresponding non-compact homogeneous spaces G/H by explicitly constructing the metric from the Maurer-Cartan form in the Iwasawa parametrization.

5.2 The gaugings

As shown by the individual analysis of appendix E, the LI forms on all cosets close under exterior differentiation into an algebra of the form (2.18) dependent on a set of parameters, the so-called geometric fluxes. Together with the Freund-Rubin parameter e_0 , these are then responsible for the gauging of the effective theory. Note that since for all our coset models the Betti number $b_3 = 0$, all the fluxes e_i, p^A and q_A are trivial. To obtain the scalar charges and the potential for each particular coset model, we just need to evaluate the general expressions in sections 3 and 4 using the specific geometric fluxes for each coset. In order to avoid overloading the paper with lengthy individual formulae, we only tabulate a consistent set of geometric fluxes for each one. These allow for the reconstruction of all relevant gauging-related quantities of the specific models using the general expressions of the previous two sections.

The set of specific geometric fluxes, $m^i, q_i^j, m_i^A, e_{iA}, u_A^B, s_{AB}, t^{AB}$, that we have brought to tables 3 and 4 are those that can be read off by comparing the general algebra (2.18) with the individual expressions given in appendix E. Although these parameters do not have an intrinsic meaning (they rather depend on the specific duality frame of the set of forms used to compute them) they are enough to characterize the charges and the scalar potential for the individual models.

We conclude this section by explicitly evaluating the scalar potentials for some of the models in table 2, using the scalar manifold parametrization of appendix E and the geometric fluxes of tables 3 and 4. Some further massaging is required to bring the potentials in the form written below. The first one is the scalar potential for the Q^{111}

M_7	m^i	q_i^j	m_i^A
$S^7 = \frac{\text{SU}(4)}{\text{SU}(3)}$	$m^1 = 2$	0	0
M^{110}	$m^1 = m^2 = 2$	0	0
Q^{111}	$m^1 = m^2 = m^3 = 2$	0	0
$V_{5,2}$	$m^1 = 2$	0	0
$S^7 = \frac{\text{Sp}(2)}{\text{Sp}(1)}$	$m^1 = m^2 = 2$	0	$m_1^0 = m_2^0 = 0$ $m_1^1 = -m_2^1 = -4$
$N(1, 1)$	$m^1 = m^2 = m^3 = 2$	0	$m_1^0 = m_2^0 = m_3^0 = 0$ $2m_1^1 = 2m_2^1 = -m_3^1 = 4$
$N(k, l)$	$m^1 = \frac{l}{2}, m^2 = \frac{k}{2},$ $m^3 = -\frac{k+l}{2}$	0	$m_1^0 = m_2^0 = m_3^0 = 1$
$N(1, -1)$	$m^1 = -\frac{1}{2}, m^2 = \frac{1}{2},$ $m^3 = m^4 = m^5 = 0$	$q_5^4 = -q_4^5 = 3,$ all other $q_i^j = 0$	$m_1^0 = m_2^0 = m_3^0 = 1,$ $m_4^0 = m_5^0 = 0$

Table 3: Geometric fluxes m^i, q_i^j and m_i^A for each of the coset models. For our choice of expansion forms, the parameters e_{iA} vanish in all models.

M_7	u_A^B	s_{AB}	t_{AB}
$S^7 = \frac{\text{SU}(4)}{\text{SU}(3)}$	0	$s_{00} = -4$	$t^{00} = 4$
M^{110}	0	$s_{00} = -4$	$t^{00} = 4$
Q^{111}	0	$s_{00} = -4$	$t^{00} = 4$
$V_{5,2}$	$u_0^0 = 0, u_0^1 = -\frac{4}{\sqrt{3}},$ $u_1^0 = \frac{4}{\sqrt{3}}, u_1^1 = 0$	$s_{00} = 0, s_{01} = 0,$ $s_{10} = 0, s_{11} = -\frac{8}{3}$	$t^{00} = 0, t^{01} = 0,$ $t^{10} = 0, t^{11} = \frac{8}{3}$
$S^7 = \frac{\text{Sp}(2)}{\text{Sp}(1)}$	0	$s_{00} = -4, s_{01} = 0,$ $s_{10} = 0, s_{11} = 0$	$t^{00} = 4, t^{01} = 0,$ $t^{10} = 0, t^{11} = 0$
$N(1, 1)$	0	$s_{00} = -4, s_{01} = 0,$ $s_{10} = 0, s_{11} = 0$	$t^{00} = 4, t^{01} = 0,$ $t^{10} = 0, t^{11} = 0$
$N(k, l)$	0	0	0
$N(1, -1)$	0	0	0

Table 4: Geometric fluxes u_A^B, s_{AB} and t_{AB} for each of the coset models.

model:

$$\begin{aligned}
V = & -8e^{2\phi}(e^{-2u_1} + e^{-2u_2} + e^{-2u_3}) + e^{4\phi}(e^{-2u_1+2u_2+2u_3} + e^{2u_1-2u_2+2u_3} + e^{2u_1+2u_2-2u_3}) \\
& + e^{4\phi-2u_1-2u_2-2u_3} \left[e^{4u_1}(b^2 + b^3)^2 + e^{4u_2}(b^1 + b^3)^2 + e^{4u_3}(b^1 + b^2)^2 \right] \\
& + \frac{1}{4}e^{4\phi-2u_1-2u_2-2u_3} \left[e_0 + 2b^1b^2 + 2b^1b^3 + 2b^2b^3 + 2(\xi^0)^2 + 2(\tilde{\xi}_0)^2 \right]^2 \\
& + 4e^{2\phi-2u_1-2u_2-2u_3} ((\xi^0)^2 + (\tilde{\xi}_0)^2),
\end{aligned} \tag{5.6}$$

where the scalars u_i , $i = 1, 2, 3$, have already been introduced above equation (5.1). Here we have set the form fluxes p^A, q_A, e_i to zero without loss of generality: we have checked that this can always be done by a field and e_0 redefinition. The potentials for the M^{110} and the $SU(4)/SU(3)$ models can be obtained from (5.6) by setting $t^3 = t^1$ and $t^3 = t^1 = t^2$ (recall that $t^i = b^i + ie^{2u_i}$). This just reflects the consistent subtruncation pattern of removing Betti vector multiplets. The potential for the $SU(4)/SU(3)$ model can be seen to coincide with that for the universal SE_7 truncation [5].

The second one is the scalar potential for the $V_{5,2}$ model, where we have also set to zero all the form fluxes, except e_0 , without loss of generality:

$$\begin{aligned}
V = & -24e^{4\phi+2u_1} - \frac{16}{3}e^{-6u_1} + \frac{1}{3} \left(9e^{2\phi+u_1} - 2e^{-\varphi-3u_1}(1 + \chi^2 + e^{2\varphi}) \right)^2 + 12e^{4\phi-2u_1}(b^1)^2 \\
& + \frac{1}{4}e^{4\phi-6u_1} \left[e_0 + 6(b^1)^2 + \frac{4}{3}(\tilde{\xi}_1)^2 + \frac{4}{\sqrt{3}}\tilde{\xi}_1\xi^0 - \frac{4}{\sqrt{3}}\tilde{\xi}_0\xi^1 + \frac{4}{3}(\xi^1)^2 \right]^2 \\
& + \frac{4}{9}e^{2\phi-6u_1} \left[3e^{3\varphi}(\xi^1)^2 + e^\varphi \left(\sqrt{3}\xi^0 + 2\tilde{\xi}_1 + 3\chi\xi^1 \right)^2 \right. \\
& \quad \left. + e^{-\varphi} \left(\sqrt{3}\tilde{\xi}_0 - 2\xi^1 + 2\chi(2\tilde{\xi}_1 + \sqrt{3}\xi^0) + 3\chi^2\xi^1 \right)^2 \right. \\
& \quad \left. + e^{-3\varphi} \left(-\sqrt{3}\tilde{\xi}_1 + \chi(3\tilde{\xi}_0 - 2\sqrt{3}\xi^1) + \chi^2(2\sqrt{3}\tilde{\xi}_1 + 3\xi^0) + \sqrt{3}\chi^3\xi^1 \right)^2 \right].
\end{aligned} \tag{5.7}$$

Here we have used, again, the scalar u_1 introduced above equation (5.1), and φ, χ in (E.45). The $V_{5,2}$ effective theory can be further truncated consistently to the universal SE_7 model, by getting rid of the additional hypermultiplet. Accordingly, the universal SE_7 potential can be recovered from (5.7) by switching off the complex structure deformations, $\varphi = 0$, $\chi = 0$, and two of the Darboux scalars $\xi^A, \tilde{\xi}_A$. Not all symplectic frames for the latter allow for the truncation to go through consistently, however: for example, it is inconsistent to set $\xi^1 = \tilde{\xi}_1 = 0$ in the frame used in (5.7). To truncate consistently to the universal model, one should first symplectically rotate

$$\begin{aligned}
\xi^0 &= \frac{1}{2}(\xi'^0 - \sqrt{3}\xi'^1), & \xi^1 &= \frac{1}{2}(-\sqrt{3}\tilde{\xi}'_0 - \tilde{\xi}'_1), \\
\tilde{\xi}_0 &= \frac{1}{2}(\tilde{\xi}'_0 - \sqrt{3}\xi'_1), & \tilde{\xi}_1 &= \frac{1}{2}(\sqrt{3}\xi'^0 + \xi'^1).
\end{aligned} \tag{5.8}$$

and then do set $\xi'^1 = \tilde{\xi}'_1 = 0$. Although no prepotential for the complex structure deformations turns out to exist in the primed basis, the latter is preferred by the mass matrix: the primed scalars are the states of definite mass (see table 7).

Finally, we provide an explicit expression for the $N(1, 1)$ potential:

$$\begin{aligned}
V = & -12e^{2\phi}(e^{-2u_1} + e^{-2u_2}) - 4e^{-2u_1-2u_2-2u_3} + e^{-2\varphi-2u_1-2u_2-2u_3}(1 + \chi^2 + e^{2\varphi})^2 \\
& + \frac{1}{2}e^{2\phi-\varphi}(1 + \chi^2 + e^{2\varphi})(e^{2u_1-2u_2-2u_3} + e^{-2u_1+2u_2-2u_3} + 4e^{-2u_1-2u_2+2u_3} - 6e^{-2u_3}) \\
& + e^{4\phi}(e^{-2u_1+2u_2+2u_3} + e^{2u_1-2u_2+2u_3} + e^{2u_1+2u_2-2u_3}) \\
& + e^{4\phi-2u_1-2u_2-2u_3}\left[e^{4u_1}(b^2 + b^3 - \tilde{\xi}_1)^2 + e^{4u_2}(b^1 + b^3 - \tilde{\xi}_1)^2 + e^{4u_3}(b^1 + b^2 + 2\tilde{\xi}_1)^2\right] \\
& + \frac{1}{4}e^{4\phi-2u_1-2u_3-2u_3}\left[e_0 + 2b^1b^2 + 2b^1b^3 + 2b^2b^3 - 2(b^1 + b^2 - 2b^3)\tilde{\xi}_1 + 2(\xi^0)^2 + 2(\tilde{\xi}_0)^2\right]^2 \\
& + \frac{1}{2}e^{2\phi-2u_1-2u_2-2u_3}\left[4e^\varphi(\xi^0)^2 + e^\varphi(b^1 + b^2 - 2b^3 - 2\tilde{\xi}_0)^2\right. \\
& \quad \left.+ e^{-\varphi}(b^1 + b^2 - 2b^3 + 2\tilde{\xi}_0 - 2\chi\xi^0)^2 + e^{-\varphi}(2\xi^0 - \chi(b^1 + b^2 - 2b^3 - 2\tilde{\xi}_0))^2\right]
\end{aligned} \tag{5.9}$$

The squashed S^7 model is obtained from here by setting $u_2 = u_1$, $b_2 = b_1$ (and then redefining $u_3 \rightarrow u_2$, $b_3 \rightarrow b_2$). The universal SE_7 model is further obtained by setting $u_3 = u_2 = u_1$, $b_3 = b_2 = b_1$, and $\tilde{\xi}_1 = \xi^1 = 0$.

5.3 Salient features

We would now like to make some further comments on each individual model, completing the description given at a glance in tables 2, 3 and 4, and the case-by-case coset construction of appendix E. In particular, we comment on sub truncation patterns and non-supersymmetric AdS vacua, and leave to next subsection the description of the supersymmetric points.

M^{110} , Q^{111} and $V_{5,2}$

The reductions on M^{110} and Q^{111} correspond to the addition of one and two Betti vector multiplets to the $SU(4)/SU(3)$ model. As pointed out in [13], the forms defining a Sasaki–Einstein structure have the same algebraic and differential properties as the LI forms on $SU(4)/SU(3)$. Accordingly, the reduction on $SU(4)/SU(3)$ can be extended to a universal reduction on any Sasaki–Einstein manifold, which coincides with that of [5]. Similarly, one can replace the \mathbb{CP}^2 in the base of M^{110} by any 4D compact Kähler–Einstein manifold and obtain a consistent reduction of the same form. The \mathbb{CP}^2 can be replaced by $\mathbb{CP}^1 \times \mathbb{CP}^1$, leading to Q^{111} , or by a del Pezzo surface.

Since all these models contain the universal SE_7 truncation, all AdS solutions of the latter are also solutions. These include the the $N = 2$ supersymmetric Sasaki–Einstein solution (and its skew-whiffed counterpart), the non-supersymmetric Pope–Warner solution [80] (which is stable within these truncations) and Englert solution [81] (which is unstable, as is already the case for the universal SE_7 truncation). This also holds for the model on $V_{5,2}$.

In addition, we find a new non-supersymmetric AdS family of solutions of the M^{110} and Q^{111} models, which are not solutions of the universal SE_7 model. It can be checked that the following,

$$\begin{aligned} e^{2U_1} &= \left(\frac{9}{5}\right)^{1/3} a, & e^{2U_2} &= \left(\frac{9}{5}\right)^{1/3} a, & e^{2U_3} &= \left(\frac{9}{5}\right)^{-2/3} a, & e^{2V} &= \frac{2}{7} 15^{2/3} a, \\ b^1 &= b^2 = b^3 = 0, & (\xi^0)^2 + (\tilde{\xi}_0)^2 &= \frac{172}{49} a^3, & e_0 &= -\frac{540}{49} a^3, \end{aligned} \quad (5.10)$$

is an AdS critical point of the Q^{111} potential (5.6), which acquires the value $V = -12\sqrt{\frac{2}{7}}a^{9/2}$. Here, $a > 0$ is just an overall scale, and the scalars U_1, U_2, U_3 are those introduced in appendix E.3. They are related to the scalars u_1, u_2, u_3 used in (5.6) via the change (E.34). Permutations of the scalars U_1, U_2, U_3 in (5.10) are also solutions. The corresponding solution of the M^{110} model follows (up to permutation) from the truncation $U_1 = U_2$, $b_1 = b_2$, consistently valid across the full moduli space. We have verified that all the masses are above the BF bound, which indicates that the solution is stable within the Q^{111} and M^{110} truncations.

Sp(2)/Sp(1) and $N(1, 1)$

These models support a tri-Sasakian structure. Moreover, it was shown in [13] that the reduction on the coset space $Sp(2)/Sp(1)$ to an $N = 4$ theory can be extended to a reduction on a generic tri-Sasakian manifold. In appendix E.5 we construct the maximal $N = 2$ subtruncation of the $N = 4$ model, which contains $n_V = 2$ vector multiplets and $n_H = 2$ hypermultiplets. Since this subtruncation can be obtained by imposing invariance under a certain \mathbb{Z}_2 symmetry on the expansion forms and since this symmetry can be expressed in terms of the tri-Sasakian structure, it follows that this subtruncation is automatically consistent and moreover that it can also be extended to a generic tri-Sasakian manifold.

One interesting point is that for every choice of metric there are two possible $SU(3)$ -structures in the model, labeled by $\epsilon = \pm 1$ in the appendix.¹¹ The $N = 2$ model therefore displays two separate branches. These have the same bosonic sector (with different range of the scalars, however), but are expected to have a different fermionic sector. Indeed, both of these branches contain all of the solutions, but since they are described with a different $SU(3)$ -structure, a different amount of supersymmetry will be visible. For instance, all of the AdS vacuum solutions of the $N = 4$ model of [13] will be present: the round solution (with manifest $N = 2$ for $\epsilon = 1$ and $N = 1$ for $\epsilon = -1$), the non-supersymmetric associated Englert solution and the Pope-Warner solution (the latter being already unstable within this truncation). Furthermore, there are the following solutions, which are rotationally symmetric in the η^I -space: the squashed solution (with $N = 0$ for $\epsilon = 1$ and $N = 1$ for $\epsilon = -1$), and the associated Englert solution.

¹¹Strictly speaking, if one goes from the $SU(3)$ -structure with $\epsilon = 1$ to the one with $\epsilon = -1$, the orientation is also changed, so that in order to obtain exactly the same solution one has to compensate this by setting $J \rightarrow -J$ and $\Omega \rightarrow \bar{\Omega}$.

We note two main subtruncations. First, one obtains the Sasaki-Einstein subtruncation of [5], with $n_V = n_H = 1$ by putting $U_1 = V_1 = V_2$ and $z = i$. As vacuum AdS solutions it contains the round solution (with $N = 2$), the associated Englert solution and the Pope-Warner solution. Second there is the truncation with $n_V = 2, n_H = 1$ that one obtains by keeping only the expansion forms that are invariant under the action of X . This amounts to putting $z = -i$. The reduced theory is equivalent to the type IIA theory of [6] on $\text{Sp}(2)/\text{S}(\text{U}(2) \times \text{U}(1))$ with Romans mass $m = 0$ [6]. As vacuum solutions, this theory has the round solution (with $N = 1$), the squashed solution and the Englert solution associated to the latter. Note that the geometric fluxes m^i that this $n_V = 2, n_H = 1$ truncation necessarily possesses, prevent a further truncation to the $n_V = 1, n_H = 1$ model corresponding to (massless) type IIA on $\text{G}_2/\text{SU}(3)$ [6] or, equivalently, IIA on any nearly-Kähler six-fold [46]. Finally, a different further subtruncation is the $N = 1$ $\text{SO}(3)_R$ -invariant truncation described in section 7.2 of [13].

Finally we note that the $N = 2$ truncation on $N(1, 1)$ can be found by adding a Betti vector multiplet to the truncation on $\text{Sp}(2)/\text{Sp}(1)$.

$N(k, l)$ and $N(1, -1)$

For generic k, l the truncation on $N(k, l)$, which has $n_V = 3, n_H = 1$, has two $N = 1$ vacuum solutions of the type of eq. (A.23). As indicated in diagram 1, these models do not have a Sasaki-Einstein subtruncation, but rather a weak G_2 -subtruncation.

A special point is $k = -l = 1$. In this case the two $N = 1$ vacuum solutions collapse to one solution. This manifold has two extra LI two-forms, so that the consistent reduction enhances to one with $n_V = 5, n_H = 1$. This allows for an extra non-supersymmetric LI Einstein solution with off-diagonal metric entries, which was first found in [82].

The model on $N(1, -1)$ is also interesting because it is the only one with non-trivial q -fluxes. According to the general analysis this implies a gauging of the scalars in the vector multiplet sector as well as the presence of a non-abelian gauge group.

5.4 The supersymmetric vacuum structure

All models admit supersymmetric AdS critical points which, by a suitable choice of geometric fluxes, can be always brought to the origin of the parametrizations of appendix E or, alternatively, arises at the points given in the appendices. The $\text{SU}(4)/\text{SU}(3)$, M^{110} , Q^{111} and $V_{5,2}$ models have only one $N = 2$ point, that renders those spaces Sasaki-Einstein. The models based on the squashed S^7 and $N(1, 1)$ possess two supersymmetric critical points, with $N = 2$ and $N = 1$. The generic $N(k, l)$ models display two $N = 1$ critical points (denoted by I and II in [36]). These coalesce for $N(1, -1)$ on a single $N = 1$ point. At the supersymmetric critical points we find that the spectrum of each model organizes itself in $\text{OSp}(4|N)$ multiplets, as it should. We have summarized the type of OSp multiplets that arise for each model at each supersymmetric point in tables 5 and 6. Appendix F

	$S^7 = \frac{\text{SU}(4)}{\text{SU}(3)}$	M^{110}	Q^{111}	V_{52}	$S^7 = \frac{\text{Sp}(2)}{\text{Sp}(1)}$ (round)	$N(1, 1)$ (round)
Gravity	1, (2, 0)	1, (2, 0)	1, (2, 0)	1, (2, 0)	1, (2, 0)	1, (2, 0)
Long vector	1, (4, 0)	1, (4, 0)	1, (4, 0)	1, (4, 0)	2, (4, 0), (3, 0)	2, (4, 0), (3, 0)
Massless vector	0	1, (1, 0)	2, (1, 0), (1, 0)	0	0	1, (1, 0)
Chiral	0	0	0	$2, (\frac{2}{3}, -\frac{2}{3}), (\frac{2}{3}, \frac{2}{3})$	0	0

Table 5: Summary of $\text{OSp}(4|2)$ multiplets at the $N = 2$ point of each model. The format of each entry is $n, (\Delta_1, R_1), \dots, (\Delta_n, R_n)$, where n is the number of multiplets of each type, and (Δ_i, R_i) are the conformal dimension and $\text{U}(1)$ R-charge of the lowest component of each multiplet.

	$S^7 = \frac{\text{Sp}(2)}{\text{Sp}(1)}$ (squashed)	$N(1, 1)$ (squashed)	$N(k, l)_{I, II}$	$N(1, -1)$
Gravity	1, (2)	1, (2)	1, (2)	1, (2)
Massive gravitino	1, (4)	1, (4)	1, (4)	1, (4)
Massive vector	0	0	0	$2, (1 + \frac{\sqrt{421}}{8})$
Massless vector	$1, (\frac{3}{2})$	$2, (\frac{3}{2}, \frac{3}{2})$	$2, (\frac{3}{2}, \frac{3}{2})$	$2, (\frac{3}{2}, \frac{3}{2})$
Chiral	$5, (5, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}, \frac{5}{3})$	$6, (5, \frac{11}{3}, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}, \frac{5}{3})$	4	4

Table 6: Summary of $\text{OSp}(4|1)$ multiplets at the $N = 1$ point of each model. The format of each entry is $n, (\Delta_1, \dots, \Delta_n)$, where n is the number of multiplets of each type, and Δ_i is the conformal dimension of the lowest component of each multiplet. The Δ_i have been omitted for the chiral multiplets of $N(k, l)$ and $N(1, -1)$.

contains the full spectra at these points. All these supersymmetric points partner with their corresponding, skew-whiffed, non-supersymmetric counterparts.

6 Outlook

In this paper, we established a general dimensional reduction of 11D supergravity on 7D manifolds with $\text{SU}(3)$ structure, leading to $N = 2$ gauged supergravity in four dimensions. We discussed how our reduction encompasses and generalizes several results previously derived in the literature on flux compactifications of 11D and type II supergravity. In the second part of the paper, we specialized to the coset manifolds $V_{5,2}$ (the Stiefel manifold), $N(k, l)$ (the Aloff–Wallach spaces), $\text{SU}(4)/\text{SU}(3)$, $\text{Sp}(2)/\text{Sp}(1)$ (which are two versions of the seven-sphere), M^{110} and Q^{111} , which represent all the homogeneous $\text{SU}(3)$ -structures admitting supersymmetric AdS_4 vacua of 11D supergravity. For each of them, we established a consistent truncation of 11D supergravity to a specific gauged $N = 2$ supergravity model, which we analysed in detail. The main features of these models having already been summarized in the Introduction, here we speculate about how our general reduction

could possibly be further extended.

We start remarking that the gaugings specified by our dimensional reduction are not the most general ones allowed by $N = 2$ supersymmetry. For instance, our vector of magnetic charges is $m^I = (0, m^i)$, implying that the magnetic dual of the graviphoton never participates in the gauging. From a purely 4D perspective, there is no obstruction against introducing a non-vanishing parameter m^0 so that m^I gets completed to $m^I = (m^0, m^i)$. Our $N = 2$ action of section 4 is already suitable for accomodating the extra charge: the only modification required is that the expression of the scalar potential gets generalized as discussed in appendix D. When the $q_i{}^j$ and \mathbb{U} charges vanish, the 11D supergravity reduction has a type IIA counterpart (see discussion at the end of section 3), and the addition of m^0 is interpreted as switching on the Romans mass. However, when a type IIA picture is not available, the higher-dimensional interpretation of m^0 is not clear. One way to incorporate it in an M-theory reduction might be to see it as a non-geometric flux: in the spirit of [75], one could generalize the first differential relation in (2.18) to $\mathcal{D}\theta = m^I\omega_I$, where \mathcal{D} is a generalized differential operator, of the form $\mathcal{D} = d + Q\cdot$, with Q being a non-geometric flux which acts by index contraction.

From a 4D point of view, there are further gaugings that could naturally be switched on. Indeed, our flux matrices transform symplectically under $\mathrm{Sp}(2n_H, \mathbb{R})$, but are symplectically incomplete as far as $\mathrm{Sp}(2n_V + 2, \mathbb{R})$ is concerned. One could speculate about a possible higher-dimensional interpretation of these extra gaugings as non-geometric fluxes (see e.g. [75] for a similar discussion in a type II framework).

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A Supersymmetry conditions for AdS₄ vacua

In this appendix we analyse the supersymmetry conditions for vacuum AdS₄ solutions to 11D supergravity, and in particular use them to show that homogeneous supersymmetric solutions are always of Freund–Rubin type. This theorem and its proof is similar to an analogous theorem for type IIA supergravity that likewise forbids static SU(2) or genuine SU(3)×SU(3)-structure supersymmetric AdS₄ solutions on homogeneous 6D compactification manifolds [83, 55]. Furthermore, we use the relation between the respective torsion classes of a G₂-structure and SU(3)-substructures to construct the supersymmetry conditions for SU(3)-structure vacua. For the manifolds studied in this paper, they lead to two important subcases.

A.1 Homogeneous supersymmetric solutions are of Freund–Rubin type

The supersymmetry conditions for a bosonic configuration in 11D supergravity are obtained by requiring that the variation of the gravitino, given by

$$\delta\Psi_M = \nabla_M\epsilon + \frac{1}{288} \left(\Gamma_M{}^{NPQR} - 8\delta_M^N \Gamma^{PQR} \right) G_{NPQR}\epsilon, \quad (\text{A.1})$$

vanishes for some 11D Majorana–Weyl spinor ϵ .

Let us first set up the reduction ansatz for vacuum solutions of the type AdS₄ × M₇, which is simpler than the off-shell reduction ansatz used elsewhere in the paper, but introduces a warp factor for completeness. For the metric we take

$$ds^2 = e^{2\Delta} g_{(4)\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n, \quad (\text{A.2})$$

where $\Delta(y)$ is the warp factor, $g_{(4)\mu\nu}$ is the AdS₄ metric and g_{mn} the metric on the internal space M₇. The ansatz for the four-form flux is

$$G_4 = e^{4\Delta} \text{vol}_4 f + F, \quad (\text{A.3})$$

where vol_4 is the volume-form built from $g_{(4)\mu\nu}$. To study the supersymmetry generator we introduce the following compactification ansatz for the gamma-matrices

$$\Gamma_\mu = e^A \gamma_\mu \otimes \mathbb{1}, \quad \Gamma_m = -i\gamma_{0123} \otimes \gamma_m, \quad (\text{A.4})$$

where γ_μ and γ_m are the 4D and 7D gamma-matrices respectively (constructed using vielbeins associated to the metrics g_{mn} and $g_{(4)\mu\nu}$ respectively). We also take $\gamma_m^\dagger = \gamma_m$ and $\gamma_{1\dots 7} = i$, where the indices in the latter and in γ_{0123} above are flat.

We will construct the supersymmetry conditions for having minimal $N = 1$ supersymmetry in 4D (which includes $N > 1$ as subcases). The general $N = 1$ ansatz for the supersymmetry generator is¹²

$$\epsilon = e^{\Delta/2} \psi_+ \otimes \zeta + e^{\Delta/2} \psi_- \otimes \zeta^c. \quad (\text{A.5})$$

¹²For an analysis of supersymmetry ansätze in this context see e.g. [57].

Here, ψ_{\pm} are 4D Weyl spinors, related by $\psi_- = D_4 \psi_+^*$, with D_4 such that $\gamma_{\mu}^* = D_4^{-1} \gamma_{\mu} D_4$. So together they define four supercharges ($N = 1$). ζ is a 7D *Dirac* spinor, with complex conjugate $\zeta^c = D_7 \zeta^*$, where D_7 is such that $\gamma_m^* = -D_7^{-1} \gamma_m D_7$.

Now we are ready to plug the compactification ansatz in (A.2). We will not aim here at a complete analysis, but mostly work towards proving the theorem stated above. First, the 4D spinors should satisfy the AdS Killing spinor equation

$$\nabla_{\mu} \psi_- = \frac{1}{2} w \gamma_{\mu} \psi_+, \quad (\text{A.6})$$

where the complex constant w is associated to the AdS radius ℓ : $|w| = 1/\ell$. Second, instead of studying the spinor equations directly it will be convenient to follow the G -structure approach and introduce differential forms constructed in terms of spinor bilinears as follows

$$\zeta^{c\dagger} \gamma_{(l)} \zeta = \frac{1}{l!} \zeta^{c\dagger} \gamma_{m_1 \dots m_l} \zeta \, dy^{m_1} \wedge \dots \wedge dy^{m_l}, \quad l = 0, 3, 4, 7. \quad (\text{A.7})$$

Using the properties under complex conjugation of the spinors and gamma-matrices one can show that the above bilinears are zero for other values of l . Moreover, we can also introduce

$$\zeta^{\dagger} \gamma_{(l)} \zeta, \quad (\text{A.8})$$

which is real for $l = 0, 1, 4, 5$ and imaginary for $l = 2, 3, 6, 7$. From the vanishing of the gravitino variation one finds the following properties for the exterior derivatives of these spinor bilinears:

$$d(e^{3\Delta} \zeta^{c\dagger} \zeta) - 2e^{2\Delta} w \zeta^{\dagger} \hat{\gamma}_{(1)} \zeta = 0, \quad (\text{A.9a})$$

$$d(e^{3\Delta} \zeta^{c\dagger} \hat{\gamma}_{(3)} \zeta) - 2e^{2\Delta} w \zeta^{\dagger} \hat{\gamma}_{(4)} \zeta + e^{3\Delta} F \zeta^{c\dagger} \zeta = 0, \quad (\text{A.9b})$$

$$d(e^{4\Delta} \zeta^{\dagger} \hat{\gamma}_{(2)} \zeta) + 3ie^{3\Delta} \text{Im}(w \zeta^{\dagger} \hat{\gamma}_{(3)} \zeta^c) - ie^{4\Delta} * F(\zeta^{\dagger} \zeta) = 0, \quad (\text{A.9c})$$

These equations can also be found from plugging our reduction ansatz in the spinor bilinear equations of [84]. For the proof of our theorem we need some more relations, which follow from the original gravitino equation, namely

$$\partial_m (e^{\Delta} \zeta^{\dagger} \hat{\gamma}^m \zeta) - \frac{5}{6} e^{\Delta} \zeta^{\dagger} F \zeta = 0, \quad (\text{A.10})$$

and

$$\frac{1}{2} e^{-\Delta} \text{Im}(w \zeta^{\dagger} \zeta^c) + \frac{f}{6} \zeta^{\dagger} \zeta = 0, \quad (\text{A.11a})$$

$$\frac{1}{2} \zeta^{\dagger} d\Delta \zeta + \frac{1}{2} e^{-\Delta} \text{Re}(w \zeta^{\dagger} \zeta^c) + \frac{1}{12} \zeta^{\dagger} F \zeta = 0. \quad (\text{A.11b})$$

In these relations, contraction of forms with gamma-matrices (the Dirac slash) is implicitly understood.

The proof now goes as follows. For a homogeneous solution all scalars have to be constant. It follows then from eq. (A.9a) that

$$\zeta^\dagger \gamma_{(1)} \zeta = 0. \quad (\text{A.12})$$

This condition is equivalent to the statement that ζ and ζ^c are proportional¹³

$$\zeta^c = e^{i\beta} \zeta, \quad (\text{A.13})$$

or alternatively the statement that $e^{i\beta/2} \zeta$ becomes a Majorana spinor. Plugging (A.12) into eqs. (A.10) and (A.11b) implies then $\text{Re}(w \zeta^\dagger \zeta^c) = 0$, or using eq. (A.13),

$$\text{Re}(w e^{i\beta}) = 0. \quad (\text{A.14})$$

Eq. (A.13) also implies $\zeta^\dagger \gamma_{(2)} \zeta = 0$, which we can use in eq. (A.9c) together with (A.14) and again (A.13) to find finally $F = 0$. From eq. (A.3) we see that this means that the flux G_4 is completely along the 4D volume, which is the hallmark of a Freund–Rubin solution. We can find the Freund–Rubin parameter f from (A.11a),

$$f = -3 e^{-\Delta} \text{Im}(w e^{i\beta}). \quad (\text{A.15})$$

Furthermore, the warp factor will be constant even for a non-homogeneous solution. This concludes the proof.

A.2 Relation between G_2 -structure and $SU(3)$ -structure

In the regime of the previous subsection, where eq. (A.13) holds, we find that there has to be a G_2 -structure on M_7 , associated to the Majorana spinor $e^{i\beta/2} \zeta$. Indeed, we can associate the only surviving non-trivial spinor bilinears to the structure form ϕ of the G_2 -structure and its Hodge dual as follows:

$$\begin{aligned} \phi &= -i \zeta^\dagger \gamma_{(3)} \zeta, \\ *_7 \phi &= \zeta^\dagger \gamma_{(4)} \zeta. \end{aligned} \quad (\text{A.16})$$

We find then from eq. (A.9b) the conditions for $N = 1$ supersymmetry imply the condition for weak G_2 holonomy:

$$d(e^{3\Delta} \phi) = 2 e^{2\Delta} \tilde{w} *_7 \phi, \quad (\text{A.17})$$

with $\tilde{w} = -i e^{i\beta} w$. It is convenient to express this in terms of the G_2 torsion classes, which correspond to the different representations of the exterior derivative of the G_2 -structure form and its Hodge dual:

$$\begin{aligned} d\phi &= \tau_0 *_7 \phi + 3 \tau_1 \wedge \phi + *_7 \tau_3, \\ d *_7 \phi &= 4 \tau_1 \wedge *_7 \phi + \tau_2 \wedge \phi, \end{aligned} \quad (\text{A.18})$$

¹³One pedestrian way to show this goes as follows. Go to an explicit representation of the 7D gamma-matrices in terms of 6D gamma-matrices. Write ζ as a sum of a 6D positive- and a negative-chirality spinor: $\zeta = \zeta_+^1 + \zeta_-^2$. Now expand $\zeta_-^2 = c \zeta_-^1 + c^i \gamma_i \zeta_+^1$. Plugging this into (A.12) one finds $c^i = 0$ and $|c|^2 = 1$, from which in the end (A.13) follows.

where τ_0, τ_1, τ_2 and τ_3 transform under G_2 as **1**, **7**, **14** and **27** respectively. We find then immediately that we must have $\tau_0 = 2\tilde{w}$ and $\tau_1 = \tau_2 = \tau_3 = 0$. In the Freund–Rubin setup, one can show that eqs. (A.15) and (A.17) (with $d\Delta = 0$) are sufficient for a supersymmetric solution.

Now, assume we have an $SU(3)$ structure on M_7 . Then the supersymmetry condition (A.17) can be translated into conditions on the $SU(3)$ -structure. In order to do this, note that having an $SU(3)$ -structure means there exists a globally defined Dirac spinor on M_7 , which yields a $U(1)$ worth of Majorana spinors. Correspondingly, we can construct a one-parameter family of G_2 -structures containing our $SU(3)$ -structure as

$$\begin{aligned}\phi &= \eta \wedge J + \text{Re}(e^{i\alpha}\Omega) , \\ \Rightarrow *_{\tau}\phi &= \frac{1}{2}J \wedge J - \eta \wedge \text{Im}(e^{i\alpha}\Omega) ,\end{aligned}\tag{A.19}$$

where the parameter α is related to the choice of the Majorana spinor defining the G_2 -structure starting from the Dirac spinor defining the $SU(3)$ -structure. The next step is to construct a relation between G_2 and $SU(3)$ -structure torsion classes. To simplify the expressions we put the vector $SU(3)$ torsion classes V_1, V_2, W_0, W_4 and W_5 , which never appear in this paper, to zero. We find then

$$\begin{aligned}\tau_0 &= \frac{1}{7} [6R + 12\text{Re}(e^{i\alpha}W_1) + 4\text{Im}E] , \quad \tau_1 = \left[\frac{1}{3}\text{Re}E + \frac{1}{2}\text{Im}(e^{i\alpha}W_1) \right] \eta , \\ \tau_2 &= T_2 + \text{Im}(e^{i\alpha}W_2) , \\ \tau_3 &= \frac{1}{14} [4R + \text{Re}(e^{i\alpha}W_1) - 2\text{Im}E] [4\eta \wedge J - 3\text{Re}(e^{i\alpha}\Omega)] \\ &\quad - \eta \wedge [T_1 + \text{Re}(e^{i\alpha}W_2)] - *_6[\text{Re}(e^{i\alpha}S) - W_3] .\end{aligned}\tag{A.20}$$

We can now analyse the supersymmetry condition (A.17) in terms of $SU(3)$ -structure.

A.3 $SU(3)$ -structure supersymmetry conditions

We are now ready to express the $N = 1$ supersymmetry condition in terms of $SU(3)$ -structure classes.¹⁴ Using (A.18) and (A.20) in (A.17) we find then (putting the warp factor to zero)

$$\begin{aligned}\tilde{w} &= \frac{1}{14} [6R + 4\text{Im}E + 12\text{Re}(e^{i\alpha}W_1)] , \quad 4R - 2\text{Im}E + \text{Re}(e^{i\alpha}W_1) = 0 , \\ T_1 + \text{Re}(e^{i\alpha}W_2) &= 0 , \quad T_2 + \text{Im}(e^{i\alpha}W_2) = 0 , \\ \text{Re}(e^{i\alpha}S) - W_3 &= 0 , \quad \frac{2}{3}\text{Re}E + \text{Im}(e^{i\alpha}W_1) = 0 .\end{aligned}\tag{A.21}$$

In our case of homogeneous structures, the last equation actually splits in $\text{Re}E = \text{Im}(e^{i\alpha}W_1) = 0$. Indeed, from (2.3) we find the relation $dJ^3 = 2\text{Re}E\eta \wedge J^3$, which, since on a compact

¹⁴For a more thorough study of the AdS_4 supersymmetry conditions on an $SU(3)$ -structure manifold, also outside the Freund–Rubin case, see [85, 57]. For a classification of $N = 2$ solutions, see [86].

manifold the volume-form cannot be exact, implies that $\text{Re} E$ cannot be everywhere non-vanishing; if the structure is homogeneous, then we conclude that $\text{Re} E = 0$. As for the solutions to the rest of the equations, we can distinguish two independent subcases, which play an important role in the paper. First let us look at solutions with $N = 2$. This means they should be valid for all α , which puts $W_1 = W_2 = S = 0$. This leads immediately to the class of Sasaki–Einstein solutions:

$$R = \tilde{w}, \quad E = 2\tilde{w}i, \quad T_1 = T_2 = W_2 = S = W_3 = 0, \quad (\text{A.22})$$

which are usually normalized by taking $\tilde{w} = 2$, in such a way that the $\text{SU}(3)$ -structure forms satisfy (2.5) and the Einstein constant of the Sasaki–Einstein metric is 6. For these solutions, the $\text{U}(1)$ generated by the vector dual to η plays the role of the R -symmetry, and the metric $ds^2(B_6)$ is Kähler–Einstein. The reduction to type IIA supergravity along this vector breaks all supersymmetry.

Second, we consider $N = 1$ solutions for which the $\text{SU}(3)$ -structure is invariant under this $\text{U}(1)$. In other words they must obey $\mathcal{L}_X J = \mathcal{L}_X \Omega = \mathcal{L}_X \eta = 0$. This leads to

$$\begin{aligned} \tilde{w} &= -3R, & \text{Re}(e^{i\alpha}W_1) + 4R &= 0, & T_1 + \text{Re}(e^{i\alpha}W_2) &= 0, \\ E &= \text{Im}(e^{i\alpha}W_1) = \text{Im}(e^{i\alpha}W_2) = W_3 = T_2 = S = 0. \end{aligned} \quad (\text{A.23})$$

Choosing for instance $e^{i\alpha} = -i$, the exterior derivatives of the $\text{SU}(3)$ -invariant forms read then as in (2.5). The solution takes the form of a $\text{U}(1)$ fibration over 6D half-flat $\text{SU}(3)$ -structure manifold. Since the structure is invariant under the vector X dual to η , reduction to type IIA along it does not break supersymmetry. This leads to the solutions of [47, 48, 49] in the limit of zero Romans mass.

Note that a priori there could also be more general $N = 1$ solutions which are not invariant under the $\text{U}(1)$, but we did not find any among the coset manifolds we considered.

B Calculation of the prepotential \mathcal{G}

At each point of a 6D $\text{SU}(3)$ -structure manifold, the space of deformations of the almost complex structure induced by the decomposable complex three-form Ω is special Kähler [59]. Sometimes this property extends globally, at least for a subset of all possible deformations. For instance, on coset manifolds this is the case as far as left-invariant deformations are concerned. In this appendix, we elaborate on the results of [59] (see also [40]) and show how the special Kähler geometry data can concretely be extracted starting from Ω . In particular, starting from an expansion of Ω in terms of a symplectic basis as in (2.6),¹⁵ we provide an algorithm for obtaining the dependent variables \mathcal{G}_B as holomorphic functions of the independent variables Z^A , as well as the prepotential $\mathcal{G}(Z)$.

¹⁵For simplicity we put in this appendix $V = 0$ in the expression (2.6) for Ω . Since turning on V only changes Ω by an overall factor, it has no effect on the expression of \mathcal{G} anyway.

In [59] it is shown that a decomposable complex three-form is completely determined by its real part through the Hitchin function. If we take $\text{Re}\Omega = x^A\alpha_A - v_A\beta^A$, then the Hitchin function is given by¹⁶

$$H(x^A, v_A) = \sqrt{-\frac{1}{6}\text{Tr } K^2}, \quad \text{with } K^m{}_n = \frac{1}{2}(\text{Re}\Omega)_{np_1p_2} \wedge \frac{1}{3!}(\text{Re}\Omega)_{p_3p_4p_5} \epsilon^{mp_1p_2p_3p_4p_5}, \quad (\text{B.1})$$

where ϵ is associated to the canonical volume-form $-\tilde{\omega}^0$. For K defined as above, one shows that $K^m{}_n K^n{}_p = c \delta^m{}_p$. Then Ω is decomposable if and only if $c < 0$, which allows for making K into an almost complex structure by normalizing. The variation of the Hitchin function is given by [59]

$$\delta H(\text{Re}\Omega) \tilde{\omega}^0 = \delta \text{Re}\Omega \wedge \text{Im}\Omega(\text{Re}\Omega). \quad (\text{B.2})$$

One finds then that $\text{Im}\Omega = y^A\alpha_A - w_A\beta^A$ is given in terms of the independent variables (x^A, v_A) in $\text{Re}\Omega$ as

$$w_A = -\frac{\partial H(x^A, v_A)}{\partial x^A}, \quad y^A = \frac{\partial H(x^A, v_A)}{\partial v_A}. \quad (\text{B.3})$$

We would however like to find a different description, where $\mathcal{G}_A = v_A + iw_A$ are the dependent variables given in terms of the independent $Z^A = x^A + iy^A$. Let us therefore define the following Legendre transform

$$\tilde{\mathcal{G}}(\text{Re } Z^A, \text{Im } Z^A) = -H(\text{Re } Z^A, \text{Re } \mathcal{G}_A(\text{Re } Z^A, \text{Im } Z^A)) + \text{Im } Z^A \text{Re } \mathcal{G}_A(\text{Re } Z^A, \text{Im } Z^A), \quad (\text{B.4})$$

where, as usual, $\text{Re } \mathcal{G}_A(\text{Re } Z^A, \text{Im } Z^A)$ is found by extremizing with respect to $\text{Re } \mathcal{G}_A$, or equivalently by solving the second equation of (B.3) for v_A . Let us remark that this is only possible if

$$\det \left(\frac{\partial^2 H}{\partial v_A \partial v_B} \right) \neq 0. \quad (\text{B.5})$$

Now $\tilde{\mathcal{G}}$ determines \mathcal{G}_A from Z^A as follows:

$$\text{Im } \mathcal{G}_A = \frac{\partial \tilde{\mathcal{G}}}{\partial (\text{Re } Z^A)}, \quad \text{Re } \mathcal{G}_A = \frac{\partial \tilde{\mathcal{G}}}{\partial (\text{Im } Z^A)}. \quad (\text{B.6})$$

$\tilde{\mathcal{G}}$ is still a real function, while what we are looking for is a holomorphic function $\mathcal{G}(Z)$ such that

$$\mathcal{G}_A = \frac{\partial \mathcal{G}}{\partial Z^A}. \quad (\text{B.7})$$

However, some calculation shows that, under the condition that (B.5) is fulfilled, $\tilde{\mathcal{G}}$ is a suitable candidate to be the imaginary part of \mathcal{G} , i.e. it satisfies Laplace's equation, which

¹⁶To be completely precise, the Hitchin function and $\text{Im}\Omega$ are only determined in terms of $\text{Re}\Omega$ up to a sign choice. The other choice of sign corresponds to plugging $H \rightarrow -H$, $\text{Im } Z^A \rightarrow -\text{Im } Z^A$, $\tilde{\mathcal{G}}(\text{Re } Z^A, \text{Im } Z^A) \rightarrow -\tilde{\mathcal{G}}(\text{Re } Z^A, -\text{Im } Z^A)$ and $\mathcal{G}(Z) \rightarrow \overline{\mathcal{G}(\overline{Z})}|_{\overline{Z} \rightarrow Z}$ in the following equations.

follows from the integrability of the Cauchy–Riemann equations, and using these Cauchy–Riemann equations (B.7) implies (B.6). We can then calculate the full holomorphic \mathcal{G} from $\text{Im}\mathcal{G}(\text{Re}Z^A, \text{Im}Z^A)$ using a formula from [87],

$$\mathcal{G}(z) - \overline{\mathcal{G}(\bar{a})} = 2i \text{Im}\mathcal{G}\left(\frac{Z^A + a}{2}, \frac{Z^A - a}{2i}\right), \quad (\text{B.8})$$

which we can evaluate for $a \rightarrow 0$. Ultimately, (B.5) becomes the condition for the existence of a holomorphic prepotential in the chosen symplectic basis.

C Details on the dimensional reduction

In this appendix we provide the details of the dimensional reduction of the 11D supergravity action (3.1) on our $\text{SU}(3)$ -structure 7D manifolds. In section C.1 we reduce the 11D Einstein–Hilbert term, while in section C.2 we turn to the form sector.

C.1 The Einstein–Hilbert term

The higher-dimensional Ricci scalar

We start by deriving a formula for the higher-dimensional Ricci scalar. The reduction ansatz is more general than in the rest of the paper, so the expression provided here can also be useful in different setups.

Suppose we have a compactification from $d+D$ to d dimensions with a certain canonical internal metric g_0 , which has L Killing vectors k_Λ ($\Lambda = 1, \dots, L$). Let us associate a vielbein e^α to the canonical metric and suppose we can choose it such that the Killing vectors are constant in flat coordinates $\partial_m k^\alpha_\Lambda = 0$. For a coset manifold left-invariant vectors will indeed give rise to such Killing vectors [88], but the result is not restricted to these manifolds. It was used for instance for generic tri-Sasakian manifolds in [13]. We take the following reduction ansatz for the metric:

$$ds^2 = e^{2\varphi} ds_d^2 + g_{\alpha\beta}(x) (e^\alpha + k^\alpha_\Lambda A^\Lambda(x)) (e^\beta + k^\beta_\Sigma A^\Sigma(x)), \quad (\text{C.1})$$

where ds_d^2 indicates an arbitrary metric in the d dimensions, and $g_{\alpha\beta}(x), A^\Lambda(x)$ are respectively scalar and vector fields in d dimensions. Furthermore, we choose φ in order to obtain the Einstein frame in d dimensions,

$$\varphi = -\frac{1}{2(d-2)} \log \frac{\det g}{\det g_0}. \quad (\text{C.2})$$

A long calculation leads to the following total Ricci scalar:

$$\begin{aligned} R = e^{-2\varphi} R_d + R_D - \frac{1}{4} e^{-4\varphi} (k_\Lambda \cdot k_\Sigma) F^\Lambda_{\mu\nu} F^{\Sigma\mu\nu} - \frac{1}{4} e^{-2\varphi} D_\mu g_{\alpha\beta} D^\mu g^{\alpha\beta} g^{\gamma\delta} \\ - 2e^{-2\varphi} \nabla^\mu \partial_\mu \varphi - (d-2) e^{-2\varphi} \partial_\mu \varphi \partial^\mu \varphi, \end{aligned} \quad (\text{C.3})$$

where R_d and R_D are the external and internal Ricci scalar (with the latter being calculated as if $g_{\alpha\beta}$ were constant and $A^\Lambda = 0$), and the d -dimensional indices μ, ν indices are raised using the metric in ds_d^2 .

Here, the covariant derivative and the field strengths are given by:

$$\begin{aligned} D_\mu g_{\alpha\beta} &= \partial_\mu g_{\alpha\beta} + 2 A_\mu^\Lambda k_\Lambda^\gamma g_{\gamma\delta} \omega^\delta_{(\alpha|m} e^m_{\beta)} = \partial_\mu g_{\alpha\beta} - (\mathcal{L}_{k_\Lambda} g)_{\alpha\beta} A_\mu^\Lambda, \\ F^\Lambda &= dA^\Lambda - \frac{1}{2} f^\Lambda_{\Sigma\Omega} A^\Sigma \wedge A^\Omega, \quad \text{where } [k_\Lambda, k_\Sigma] = f^\Omega_{\Lambda\Sigma} k_\Omega. \end{aligned} \quad (\text{C.4})$$

Here ω^α_β are the spin connection one-forms associated to the coframe e^α . Note that the covariant derivative reduces to the ordinary derivative if k_Λ is a Killing vector of the full metric $g_{\alpha\beta}$ (as opposed to only of the canonical metric).

In this paper we have $d = 4, D = 7$, and there is only one Killing vector, k . This is the dual of the one-form θ , and is related to the defining vector of the $\text{SU}(3)$ -structure X as $k = e^V X$. Our canonical metric has volume form $-\tilde{\omega}^0 \wedge \theta$, so for the Weyl factor φ we find

$$e^{2\varphi} = \frac{\sqrt{\det g_{0(D)}}}{\sqrt{\det g_{(D)}}} = \frac{-\tilde{\omega}^0 \wedge \theta}{\frac{1}{6} J \wedge J \wedge J \wedge \eta} = e^{2V} \mathcal{K}^{-1}. \quad (\text{C.5})$$

Then the metric ansatz (C.1) reduces to the one given in the main text in (3.2). We will look at calculating the different parts of expression (C.3) next.

The internal Ricci scalar

One of the ingredients in the expression (C.3) is the internal Ricci scalar R_7 , which we would like to express in terms of $\text{SU}(3)$ torsion classes, and ultimately in terms of 4D scalar fields and geometric fluxes. Taking our cue from [89] we will make use of the fact that the 7D Ricci scalar has already been calculated in terms of G_2 torsion classes [90] and there exists a one-parameter family of G_2 -structures containing our $\text{SU}(3)$ -structure (and thus leading to the same metric), given by (A.19). In terms of the G_2 torsion classes, the Ricci scalar is given by [90]:

$$R_7 = -12 *_{\eta} d *_{\eta} \tau_1 + \frac{21}{8} \tau_0^2 + 30(\tau_1)^2 - \frac{1}{2}(\tau_2)^2 - \frac{1}{2}(\tau_3)^2. \quad (\text{C.6})$$

Plugging the relation (A.20) between G_2 - and $\text{SU}(3)$ -structure classes into the above expression, we find

$$\begin{aligned} R_7 &= -4 \eta \lrcorner d(\text{Re } E) + \frac{15}{2} |W_1|^2 - \frac{3}{2} R^2 - \frac{14}{3} (\text{Re } E)^2 + 6R(\text{Im } E) \\ &\quad - \frac{1}{2} (T_2)^2 - \frac{1}{2} |W_2|^2 - \frac{1}{2} (T_1)^2 - \frac{1}{2} (W_3)^2 - \frac{1}{4} |S|^2. \end{aligned} \quad (\text{C.7})$$

In order to simplify the expression, we used the following quadratic constraints on the torsion classes:

$$\begin{aligned} 6 W_1 R - T_1 \cdot W_2 &= 0, \\ 6 \eta \lrcorner dW_1 - 2W_1 E + 4W_1 \bar{E} + W_2 \cdot T_2 - iS \cdot W_3 &= 0, \end{aligned} \quad (\text{C.8})$$

which can be obtained by acting with an exterior derivative on (2.3) and imposing consistency. Note furthermore that the above expression for the Ricci scalar is manifestly invariant under the $U(1)$ rotation induced by shifting α in (A.19) (as it should be since the metric is also invariant).

To reexpress (C.7) in terms of the geometric fluxes, it turns out to be convenient to first rewrite it as

$$R_7 = *_7 \left[\frac{1}{2}(\eta \lrcorner dJ) \wedge (\eta \lrcorner dJ) \wedge J + \frac{i}{4}(\eta \lrcorner d\Omega) \wedge (\eta \lrcorner d\bar{\Omega}) + \frac{1}{2} d\eta \wedge d\eta \wedge J \right] - \frac{1}{2}|d\Omega|_{\eta=0}|^2 - \frac{1}{2}|dJ|_{\eta=0}|^2 - 8(\text{Re } E)^2 - 2(\text{Im } E)^2 + 6R \text{Im } E - \frac{9}{2}R^2 + 18|W_1|^2. \quad (\text{C.9})$$

Plugging in the expressions for the scalar torsion classes (2.22) and further working out the derivatives on J and Ω in terms of geometric fluxes by using the expansions in section 2.2, we obtain in the end the expression given in (C.23) below.

The kinetic terms

The kinetic terms in the effective action come from the third to last term in eq. (C.3) (with the proper prefactor $\frac{1}{2}e^{2\varphi}$). Recalling that in the metric ansatz (3.2) we have just one vector, k , and that the 4D gauge vector is called A^0 , from the third term we find immediately the gauge kinetic term

$$-\frac{1}{8}e^{-2\varphi}(k_\Lambda \cdot k_\Sigma)F_{\mu\nu}^\Lambda F^{\Sigma\mu\nu} = -\frac{1}{4}\mathcal{K}(dA^0)_{\mu\nu}(dA^0)^{\mu\nu}. \quad (\text{C.10})$$

Next we will study the 4D kinetic terms for the scalars from the internal metric by working out the fourth term in (C.3),

$$-\frac{1}{8}D_\mu g_{\alpha\beta} D^\mu g_{\gamma\delta} g^{\alpha\gamma} g^{\beta\delta}, \quad (\text{C.11})$$

in terms of J , Ω and η . Let us first construct the action of the covariant derivative on these forms, starting from its action (C.4) on the metric. Again setting the vector $SU(3)$ torsion classes to zero, we obtain from (2.3)

$$\begin{aligned} D_\mu \eta &= \partial_\mu \eta - (\mathcal{L}_k \eta) A_\mu^0 = \partial_\mu \eta, \\ D_\mu J &= \partial_\mu J - (\mathcal{L}_k J) A_\mu^0 = \partial_\mu J - e^V \left(\frac{2}{3} \text{Re } E J + T_2 \right) A_\mu^0, \\ D_\mu \Omega &= \partial_\mu \Omega - (\mathcal{L}_k \Omega) A_\mu^0 = \partial_\mu \Omega - e^V (E \Omega + S) A_\mu^0. \end{aligned} \quad (\text{C.12})$$

Plugging in eqs. (2.6) and (2.18), the Lie derivatives of the forms can be computed in terms of the metric moduli and of the geometric fluxes, and we find the expressions for Dv^i , DZ^A , DG_A given in the main text in eq. (3.16). Note that the transformation of Z , which A_μ^0 gauges,

$$\delta \Omega = -e^V (E \Omega + S), \quad (\text{C.13})$$

is a proper complex deformation since S is a (2,1)-form. This implies that the functional form of the $\mathcal{G}_A(Z)$ does not change.

The treatment of (C.11) is standard, see e.g. [60]. Although [60] considered Calabi–Yau manifolds, their analysis of the kinetic terms (C.11) should also apply to our case, since it does not depend on internal exterior derivatives. However, our formulae are going to work even without integrating on the compact manifold; this guarantees that the reduction goes through not only at the level of the action, but also at the level of the equations of motion. Working out the kinetic terms we find

$$\begin{aligned} & -\frac{1}{8}D_\mu g_{\alpha\beta} D^\mu g_{\gamma\delta} g^{\alpha\gamma} g^{\beta\delta} = \\ & -\frac{1}{2}\partial_\mu V \partial^\mu V - \frac{1}{4}\text{Tr}(D_\mu g)_{(2,0)} g^{-1} (D^\mu g)_{(0,2)} g^{-1} - \frac{1}{8}\text{Tr}(D_\mu g)_{(1,1)} g^{-1} (D^\mu g)_{(1,1)} g^{-1}. \end{aligned} \quad (\text{C.14})$$

Here $(D^\mu g)_{(2,0)+(0,2)}$ is associated to the deformations of Ω , while $(D^\mu g)_{(1,1)}$ is associated to the deformation of J .

Let us start with the deformations of Ω . Following [60] we define the (2,1)-forms χ_a as

$$\frac{\partial \Omega}{\partial z^a} = \kappa_a \Omega + \chi_a. \quad (\text{C.15})$$

One can then show, analogously to [60], that

$$(D_\mu g)_{(0,2)\alpha\beta} = \frac{1}{8} \chi_{a\alpha\gamma\delta} \bar{\Omega}_\beta{}^{\gamma\delta} D_\mu z^a, \quad (\text{C.16})$$

and finally

$$-\frac{1}{4}\text{Tr}(D_\mu g)_{(2,0)} g^{-1} (D^\mu g)_{(0,2)} g^{-1} = -\frac{1}{8} \chi_a \cdot \bar{\chi}_b D_\mu z^a D^\mu \bar{z}^b = -g_{a\bar{b}} D_\mu z^a D^\mu \bar{z}^b, \quad (\text{C.17})$$

where $g_{a\bar{b}}$ is the special Kähler metric associated to the Kähler potential K_Ω defined in (2.23).

Turning to the (1,1)-part we find:

$$\begin{aligned} -\frac{1}{8}\text{Tr}(D_\mu g)_{(1,1)} g^{-1} (D^\mu g)_{(1,1)} g^{-1} &= -\frac{1}{4} D_\mu J \cdot D^\mu J = -\frac{1}{4} *_7 (D_\mu J \wedge *_7 D^\mu J) \\ &= -\frac{3}{4} \partial_\mu V \partial^\mu V + \frac{1}{2\mathcal{K}} \partial_\mu V D^\mu \mathcal{K} - g_{ij} D_\mu v^i D^\mu v^j, \end{aligned} \quad (\text{C.18})$$

where to write the last line we used the expansion of J introduced in (2.6) and the definition (2.26) of the metric g_{ij} .

Eventually for the scalar kinetic terms we arrive at

$$\frac{1}{8} g^{\alpha\gamma} g^{\beta\delta} D_\mu g_{\alpha\beta} D^\mu g_{\gamma\delta} + \partial_\mu \varphi \partial^\mu \varphi = \partial_\mu \phi \partial^\mu \phi + g_{ij} D_\mu v^i D^\mu v^j + g_{a\bar{b}} D_\mu z^a D^\mu \bar{z}^b, \quad (\text{C.19})$$

where we introduced the 4D dilaton

$$e^{2\phi} := e^{3V} \mathcal{K}^{-1} = e^{2\varphi+V}. \quad (\text{C.20})$$

The 4D action

Using the above results, the 11D Einstein–Hilbert term reduces to the following 4D action:

$$S_{\text{EH}} = \frac{1}{\kappa_4^2} \int \left(\frac{1}{2} R_4 * 1 - \frac{1}{4} \mathcal{K} dA^0 \wedge * dA^0 + g_{ij} Dv^i \wedge * Dv^j + g_{a\bar{b}} Dz^a \wedge * D\bar{z}^{\bar{b}} + d\phi \wedge * d\phi - V_{\text{geo}} * 1 \right), \quad (\text{C.21})$$

where $\kappa_4^{-2} = \kappa_{11}^{-2} \int (-\tilde{\omega}^0 \wedge \theta)$, and the Hodge star is with respect to the 4D metric ds_4^2 . The contribution from the 11D Einstein–Hilbert term to the 4D scalar potential, V_{geo} , follows from the internal Ricci scalar R_7 as

$$V_{\text{geo}} = -\frac{1}{2} e^{2V} \mathcal{K}^{-1} R_7, \quad (\text{C.22})$$

and is found to be

$$\begin{aligned} V_{\text{geo}} = & 8 e^K g_{ij} v^k q_k^i v^l q_l^j + \frac{1}{8} e^{-K+4\phi} g_{ij} m^i m^j - 2 e^{K+2\phi} v^i v^j Q_i^T \mathbb{C} \mathbb{M} Q_j \\ & + 4 e^{K+K_\Omega+2\phi} (g^{ij} - 4 v^i v^j) (Q_i^T \mathbb{C} Z) (Q_j^T \mathbb{C} \bar{Z}) \\ & + 8 i e^{K+K_\Omega} Z^T \mathbb{U}^T \mathbb{C} \mathbb{U} \bar{Z} + 8 e^{K+2K_\Omega} (Z^T \mathbb{C} \mathbb{U} \bar{Z})^2 + 4 e^{K+K_\Omega+2\phi} (m^i \mathcal{K}_i) (Z^T \mathbb{C} \mathbb{U} \bar{Z}). \end{aligned} \quad (\text{C.23})$$

C.2 The form sector

In this section we describe the reduction of the kinetic and Chern–Simons terms of the 11D supergravity three-form. This is done by plugging the expansions (3.3)–(3.6) in the 11D action (3.1). We also checked the result by reducing the 11D equations of motion and reconstructing the corresponding 4D action.

By evaluating the exterior derivative of the 11D three-form A_3 , expanded as in (3.3), we find that the field strength (3.5), (3.6) is

$$\begin{aligned} G_4 = dA_3 + G_4^{\text{flux}} = & H_4 + dB \wedge (\theta + A^0) + H_2^i \wedge \omega_i + Db^i \wedge \omega_i \wedge (\theta + A^0) \\ & + D\xi^A \wedge \alpha_A - D\tilde{\xi}_A \wedge \beta^A + \chi_i \tilde{\omega}^i \\ & + [(b^I Q_I + \mathbb{U}\xi)^A \alpha_A - (b^I Q_I + \mathbb{U}\xi)_A \beta^A] \wedge (\theta + A^0), \end{aligned} \quad (\text{C.24})$$

where we introduced the following combinations of 4D fields and charges:

$$\begin{aligned} H_4 &= dC_3 + B \wedge dA^0, \\ H_2^i &= -dA^i + A^0 \wedge A^j q_j^i + m^i B + b^i dA^0, \\ \chi_i &= e_i + \mathcal{K}_{ijk} m^j b^k + Q_i^T \mathbb{C} \xi, \end{aligned} \quad (\text{C.25})$$

together with $b^I \equiv \text{Re} X^I = (1, b^i)$, and with the scalar derivatives given in (3.18). As already mentioned in the main text, we dualize the three-form C_3 to a constant e_0 , following e.g. [70]; this leads to

$$H_4 = \mathcal{K}^{-1} e^{4\phi} (b^I \mathcal{E}_I + \frac{1}{2} \mathcal{K}_{ijk} m^i b^j b^k) * 1, \quad (\text{C.26})$$

where we introduced the combination

$$\mathcal{E}_I = e_I + Q_I^T \mathbb{C} \xi - \frac{1}{2} \delta_I^0 \xi^T \mathbb{C} \mathbb{U} \xi. \quad (\text{C.27})$$

In the end we obtain the 4D action

$$S_{G_4} = \frac{1}{\kappa_4^2} \int [\mathcal{L}_{\text{kin}, G_4} + \mathcal{L}_{\text{top}} - V_{G_4} * 1], \quad (\text{C.28})$$

where the kinetic plus instanton contributions are

$$\begin{aligned} \mathcal{L}_{\text{kin}, G_4} = & -\frac{1}{4} \mathcal{K}_{ijk} b^i H_2^j \wedge H_2^k + \frac{1}{4} \mathcal{K}_{ijk} b^i b^j H_2^k \wedge dA^0 - \frac{1}{12} \mathcal{K}_{ijk} b^i b^j b^k dA^0 \wedge dA^0 \\ & + \frac{1}{4} e^{-4\phi} dB \wedge *dB - \mathcal{K} g_{ij} H_2^i \wedge *H_2^j + g_{ij} D b^i \wedge *D b^j - \frac{1}{4} e^{2\phi} (D\xi)^T \wedge *(\mathbb{C} \mathbb{M} D\xi), \end{aligned} \quad (\text{C.29})$$

while the other topological terms read

$$\begin{aligned} \mathcal{L}_{\text{top}} = & -\frac{1}{4} dB \wedge [(2e_I + Q_I^T \mathbb{C} \xi) A^I - \xi^T \mathbb{C} D\xi] \\ & + \frac{1}{4} q_i^I \mathcal{K}_{jkl} A^i \wedge A^j \wedge (dA^k - m^k B) - \frac{1}{4} e_i m^i B \wedge B, \end{aligned} \quad (\text{C.30})$$

and the scalar potential is

$$\begin{aligned} V_{G_4} = & -\frac{1}{4} e^{2\phi} \mathcal{K}^{-1} (b^I Q_I^T + \xi^T \mathbb{U}^T) \mathbb{C} \mathbb{M} (b^J Q_J + \mathbb{U} \xi) \\ & + \frac{1}{4} e^{4\phi} \mathcal{K}^{-1} (b^I \mathcal{E}_I + \frac{1}{2} \mathcal{K}_{ijk} m^i b^j b^k)^2 + \frac{1}{16} e^{4\phi} \mathcal{K}^{-1} g^{ij} \chi_i \chi_j. \end{aligned} \quad (\text{C.31})$$

We can now recast some of the above expressions in a form that is suitable for matching the $N = 2$ supergravity formalism. In terms of the field strengths defined in (3.14), we have

$$H_2^i = -F^i + b^i F^0. \quad (\text{C.32})$$

Using this, one can rewrite the gauge kinetic and topological terms in (C.29) as

$$-\frac{1}{4} \mathcal{K}_{ijk} b^i H_2^j \wedge H_2^k + \frac{1}{4} \mathcal{K}_{ijk} b^i b^j H_2^k \wedge dA^0 - \frac{1}{12} \mathcal{K}_{ijk} b^i b^j b^k dA^0 \wedge dA^0 = \frac{1}{4} \text{Re} \mathcal{N}_{IJ} F^I \wedge F^J \quad (\text{C.33})$$

and

$$-\mathcal{K} g_{ij} H_2^i \wedge *H_2^j = \frac{1}{4} \text{Im} \mathcal{N}_{IJ} F^I \wedge *F^J + \frac{1}{4} \mathcal{K} F^0 \wedge *F^0, \quad (\text{C.34})$$

where the gauge kinetic matrix \mathcal{N}_{IJ} is precisely the special Kähler geometry period matrix (2.33) obtained from the prepotential (2.31). Using the same matrix, and recalling (2.30) as well as the last in (C.25), we can also rewrite V_{G_4} in (C.31) as

$$\begin{aligned} V_{G_4} = & -2 e^{K+2\phi} (b^I Q_I^T + \xi^T \mathbb{U}^T) \mathbb{C} \mathbb{M} (b^J Q_J + \mathbb{U} \xi) \\ & - \frac{1}{4} e^{4\phi} (\mathcal{E}_I - \text{Re} \mathcal{N}_{IK} m^K) (\text{Im} \mathcal{N})^{-1IJ} (\mathcal{E}_J - \text{Re} \mathcal{N}_{JL} m^L). \end{aligned} \quad (\text{C.35})$$

Note that this is positive-definite, because both $\text{Im} \mathcal{N}$ and $\mathbb{C} \mathbb{M}$ are negative-definite.

After these reformulations are implemented, and the contribution (C.21) from the 11D Einstein–Hilbert term derived above is added, the 4D action takes exactly the form given in section 3.2.

D Matching the $N = 2$ scalar potential

In this section, we show that the scalar potential determined by the $N = 2$ gauging described in section 4 matches the expression (3.21) obtained from the dimensional reduction.

The general form of the scalar potential in 4D $N = 2$ gauged supergravity for an electric-magnetic gauging of the type (4.7) is [66, 69, 71, 72, 67]:

$$V = 2 \left\{ e^K g_{kl} k_I^k \bar{k}_J^l \bar{X}^I X^J + 4 e^K h_{uv} k^u \bar{k}^v - \left[\frac{1}{2} (\text{Im } \mathcal{N})^{-1IJ} + 4 e^K X^I \bar{X}^J \right] (\mathcal{P}_I^x - \mathcal{N}_{IK} \tilde{\mathcal{P}}^{xK}) (\mathcal{P}_J^x - \bar{\mathcal{N}}_{JL} \tilde{\mathcal{P}}^{xL}) \right\}, \quad (\text{D.1})$$

where we used the symplectic invariant combination $k^u = X^I k_I^u - \mathcal{F}_I \tilde{k}^{Iu}$, and the overall factor of 2 is due to the fact that our Killing vectors and Killing prepotentials are rescaled by a factor of $\sqrt{2}$ with respect to the ones in e.g. [66], see footnote 10.

We now plug in the $N = 2$ data found in section 4. With no extra effort, we can perform the computation by allowing for an arbitrary flux charge m^0 , so that $m^I = (m^0, m^i)$. This vanishes in 11D supergravity, while it corresponds to the Romans mass in type IIA.

We find that the contribution from the vector multiplet gauging is

$$2 e^K g_{k\bar{l}} k_I^k \bar{k}_J^{\bar{l}} \bar{X}^I X^J = 8 e^K g_{ij} v^k q_k^i v^l q_l^j, \quad (\text{D.2})$$

while for the hypermultiplet gauging we get from the first line of (D.1)

$$\begin{aligned} 8 e^K h_{uv} k^u \bar{k}^v &= 8 e^K g_{a\bar{b}} k^a \bar{k}^{\bar{b}} - 2 e^{K+2\phi} (X^I Q_I^T + \xi^T \mathbb{U}^T) \mathbb{C} \mathbb{M} (Q_J \bar{X}^J + \mathbb{U} \xi) \\ &\quad + 2 e^{K+4\phi} |X^I e_I + X^I Q_I^T \mathbb{C} \xi - \mathcal{F}_I m^I - \frac{1}{2} \xi^T \mathbb{C} \mathbb{U} \xi|^2, \end{aligned} \quad (\text{D.3})$$

where $g_{a\bar{b}} k^a \bar{k}^{\bar{b}}$ still needs to be evaluated. By expressing $g_{a\bar{b}}$ in terms of the derivatives of the Kähler potential $K_\Omega = -\log[i(\bar{Z}^A \mathcal{G}_A - Z^A \bar{\mathcal{G}}_A)]$, and identifying that $z^a = Z^a/Z^0$, one derives the following relation:

$$g_{ab} \delta z^a \delta \bar{z}^b = e^{K_\Omega} (\text{Im } \mathcal{G}_{AB} + 2 e^{K_\Omega} \text{Im } \mathcal{G}_{AC} \bar{Z}^C \text{Im } \mathcal{G}_{BD} Z^D) \delta Z^A \delta \bar{Z}^B. \quad (\text{D.4})$$

For a variation as in (4.11), recalling the consistency condition (3.17), we obtain then

$$g_{a\bar{b}} k^a \bar{k}^{\bar{b}} = i e^{K_\Omega} Z^T \mathbb{U}^T \mathbb{C} \mathbb{U} \bar{Z} + e^{2K_\Omega} (\bar{Z}^T \mathbb{C} \mathbb{U} Z)^2. \quad (\text{D.5})$$

Finally, (2 times) the second line of (D.1) gives the expression

$$\begin{aligned} &4 e^{K+K_\Omega+2\phi} (g^{ij} - 4 v^i v^j) (Z^T \mathbb{C} Q_i) (\bar{Z}^T \mathbb{C} Q_j) \\ &- \frac{1}{4} e^{4\phi} (\text{Im } \mathcal{N})^{-1IJ} (\mathcal{E}_I - \mathcal{N}_{IK} m^K) (\mathcal{E}_J - \bar{\mathcal{N}}_{JL} m^L) \\ &- 2 e^{K+4\phi} |X^I e_I + X^I Q_I^T \mathbb{C} \xi - \mathcal{F}_I m^I - \frac{1}{2} \xi^T \mathbb{C} \mathbb{U} \xi|^2 \\ &+ 4 e^{K+K_\Omega+2\phi} (m^i \mathcal{K}_i - m^0 b^i \mathcal{K}_i) (Z^T \mathbb{C} \mathbb{U} \bar{Z}), \end{aligned} \quad (\text{D.6})$$

with \mathcal{E}_I as in (3.22), and where in evaluating the $x = 1, 2$ contribution (yielding the first line in (D.6)) it was useful to notice that for cubic prepotentials one has

$$\frac{1}{2}(\text{Im}\mathcal{N})^{-1IJ} + 2e^K(X^I\bar{X}^J + \bar{X}^IX^J) = -\delta_k^I\delta_l^J e^K(g^{kl} - 4v^kv^l). \quad (\text{D.7})$$

Adding everything up, and setting $m^0 = 0$, we obtain precisely the expression for V in (3.21).

E The cosets

Here we would like to collect several details about the construction of the left-invariant forms of the compact homogeneous spaces that we consider in this paper. For each coset, we also construct the corresponding LI $\text{SU}(3)$ -structure and its associated metric. Although some of this data have already appeared in the literature, we present here a self-contained summary.

Recall that, a global, left (say) action of the group G is defined on a homogeneous manifold $G/H = \{gH : g \in G\}$. This generates motions on the manifold along which a frame is rotated. Tensors that are invariant under this action are called *left-invariant* (LI) tensors. These are globally-defined, and have the property that their value at any point is determined by the value they take at the origin of the group. Given a D -dimensional homogeneous space, one can then study which G -structures can be constructed using LI forms. This is determined by the embedding of H in $\text{SO}(D)$. In this section, we are going to study 7D homogeneous spaces which admit left-invariant $\text{SU}(3)$ -structures. Within our assumptions, these are the spaces collected in table 1.

We start by briefly reviewing the notion of left-invariance on coset manifolds G/H (for more details see e.g. [91]). On a patch with coordinates y , take a coset representative $L(y)$. Let $\{\mathcal{H}_\sigma\}$ be a basis of generators of the algebra of H , and $\{\mathcal{K}_\alpha\}$ a basis of the complement of that algebra in the algebra of G . The decomposition of the Lie-algebra valued one-form

$$L^{-1}dL = e^\alpha\mathcal{K}_\alpha + \omega^\sigma\mathcal{H}_\sigma, \quad (\text{E.1})$$

defines a local coframe $e^\alpha(y)$ on G/H . Furthermore, in terms of the structure constants f of G one has

$$de^\alpha = -\frac{1}{2}f^\alpha_{\beta\gamma}e^\beta \wedge e^\gamma - f^\alpha_{\sigma\beta}\omega^\sigma \wedge e^\beta. \quad (\text{E.2})$$

One can show that a p -form ϕ is left-invariant if and only if it can be written as

$$\phi = \frac{1}{p!}\phi_{\alpha_1\dots\alpha_p}e^{\alpha_1} \wedge \dots \wedge e^{\alpha_p}, \quad (\text{E.3})$$

with the components $\phi_{\alpha_1\dots\alpha_p}$ being independent of y , and satisfying

$$f^\beta_{\sigma[\alpha_1}\phi_{\alpha_2\dots\alpha_p]\beta} = 0. \quad (\text{E.4})$$

The left-invariance property is preserved by the exterior derivative. One can also show that harmonic forms must be left-invariant and thus the cohomology of the coset manifold is isomorphic to the cohomology of left-invariant forms. The left-invariance condition for a metric on the coset is similar, namely it must be of the form

$$ds_{G/H}^2 = g_{\alpha\beta} e^\alpha e^\beta, \quad (\text{E.5})$$

with $g_{\alpha\beta}$ being independent of y , and satisfying

$$f^\gamma{}_{\sigma(\alpha} g_{\beta)\gamma} = 0. \quad (\text{E.6})$$

We will now use this technology to determine the LI forms on each coset of table 1.

E.1 $S^7 = \text{SU}(4)/\text{SU}(3)$

We start by introducing a basis for the fifteen generators of $\text{SU}(4)$ in the fundamental representation:

$$T_i = -\frac{i}{2}\mu_i, \quad i = 1, \dots, 7, \quad T_{7+a} = -\frac{i}{2} \begin{pmatrix} \lambda_a & 0 \\ 0 & 0 \end{pmatrix}, \quad a = 1, \dots, 8, \quad (\text{E.7})$$

where λ_a , $a = 1, \dots, 8$, are the Gell-Mann matrices, and

$$\begin{aligned} \mu_1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ \mu_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \mu_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mu_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \\ \mu_7 &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \end{aligned} \quad (\text{E.8})$$

The generators that exponentiate into the coset and the generators of $H = \text{SU}(3)$ are taken to be respectively

$$\{\mathcal{K}_\alpha\} = \{T_1, \dots, T_7\}, \quad \alpha = 1, \dots, 7, \quad \{\mathcal{H}_\sigma\} = \{T_8, \dots, T_{15}\}, \quad \sigma = 1, \dots, 8. \quad (\text{E.9})$$

We can now use this representation to find the structure constants of $\text{SU}(4)$, and scan for all possible LI forms using eq. (E.4). We then use them to construct the following set

of expansion forms, which introducing the vector k dual to θ can be seen to satisfy the algebraic restrictions (2.7)–(2.9)

$$\begin{aligned}\theta &\equiv -\sqrt{\frac{3}{8}}e^7 \\ \omega_1 &\equiv \frac{1}{4}(e^{12} + e^{34} + e^{56}) , \quad \tilde{\omega}^1 \equiv \frac{1}{48}(e^{3456} + e^{1256} + e^{1234}) , \quad \tilde{\omega}^0 \equiv -\frac{1}{64}e^{123456} , \\ \alpha_0 &\equiv \frac{1}{16}(e^{135} - e^{146} - e^{236} - e^{245}) , \quad \beta^0 \equiv -\frac{1}{16}(e^{246} - e^{136} - e^{145} - e^{235})\end{aligned}\quad (\text{E.10})$$

Furthermore they satisfy eq. (2.13) with

$$\mathcal{K}_{111} = 6 . \quad (\text{E.11})$$

Further, using (E.2), these forms can be further checked to close into the following exterior algebra:

$$d\theta = 2\omega , \quad d\omega_1 = d\tilde{\omega}^1 = d\tilde{\omega}^0 = 0 , \quad d\alpha_0 = -4\theta \wedge \beta^0 , \quad d\beta^0 = 4\theta \wedge \alpha_0 , \quad (\text{E.12})$$

from which we immediately read off the geometric fluxes of tables 3 and 4.

We can now use these forms to expand the SU(3)-structure on SU(4)/SU(3) as

$$\eta = e^V \theta , \quad J = e^{2U_1} \omega , \quad \Omega = 2 e^{3U_1} (\alpha_0 + i\beta^0) , \quad (\text{E.13})$$

where U_1, V are two arbitrary constants, that become scalars in the effective theory. In order to make contact with the main text notation, these may be alternatively reparametrized in terms of two other scalars u_1, ϕ as

$$u_1 = U_1 + \frac{1}{2}V , \quad \phi = -3U_1 . \quad (\text{E.14})$$

The scalar ϕ corresponds to the 4D dilaton, eq. (3.20), while, in the notation of eq. (2.6), $v^1 \equiv e^{2u_1}$ lies in a vector multiplet. Note that, in this and all models with $n_H = 1$, only the overall scale of Ω can change, which leads to a trivial moduli space of almost complex structures. Moreover, we fix the scale of Ω through the normalization below (2.1). Finally, the torsion classes of the SU(3)-structure (E.13) can be calculated. In the notation of (2.3), the only active classes in this model are the scalars R and E . We omit the details here: see eq. (E.35) for Q^{111} from where these can be easily deduced.

Finally, the LI metric on SU(4)/SU(3) associated to the SU(3)-structure (E.13) reads

$$ds_7^2 = \frac{1}{4}e^{2U_1} [(e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 + (e^5)^2 + (e^6)^2] + e^{2V}\theta^2 . \quad (\text{E.15})$$

For $U_1 = V = 0$ this metric is SE₇, canonically normalized such that $R_{mn} = 6g_{mn}$. We omit the details here, and refer to section E.3 for a similar derivation.

Finally, as already noted in [13], the truncation associated to SU(4)/SU(3) is identical to the universal SE₇ truncation of [5]. The scalars U_1, V are related to the breathing and squashing modes of [26, 5]. Here they are clearly seen to independently rescale the fiber and the base $B_6 = \mathbb{CP}^3$ of $S^7 = \text{SU}(4)/\text{SU}(3)$. The truncation does not depend on the details of \mathbb{CP}^3 , which can be thus be replaced by any Kähler–Einstein six-fold, leading to the universal SE₇ truncation.

E.2 M^{110}

M^{pq0} is the coset G/H with $G = \text{SU}(3) \times \text{SU}(2)$ and $H = \text{SU}(2) \times \text{U}(1)$, with $\text{SU}(2)$ embedded into $\text{SU}(3)$ as the isospin subgroup, and p, q determining the embedding of the Cartan subgroup $\text{U}(1)^2$ of H into G . We can make this more precise by introducing the eleven generators of G in terms of Gell-Mann and Pauli matrices in the following block diagonal form, where the first block in the diagonal is 3-by-3 and the second 2-by-2:

$$T_a = -\frac{i}{2} \begin{pmatrix} \lambda_a & 0 \\ 0 & 0 \end{pmatrix}, \quad T_{8+i} = -\frac{i}{2} \begin{pmatrix} 0 & 0 \\ 0 & \sigma_i \end{pmatrix}. \quad (\text{E.16})$$

We take the seven generators that exponentiate into the coset M^{pq0} as

$$\{\mathcal{K}_\alpha\} = \{T_4, T_5, T_6, T_7, T_9, T_{10}, Z_1 \equiv \frac{1}{\sqrt{3p^2+q^2}} (p\sqrt{3}T_8 + qT_{11})\}, \quad \alpha = 1, \dots, 7, \quad (\text{E.17})$$

while those defining $H = \text{SU}(2) \times \text{U}(1)$ as

$$\{\mathcal{H}_\sigma\} = \{T_1, T_2, T_3, Z_2 \equiv \frac{1}{\sqrt{3p^2+q^2}} (-qT_8 + p\sqrt{3}T_{11})\}, \quad \sigma = 1, \dots, 4. \quad (\text{E.18})$$

We can now determine the LI forms supported by this coset, proceeding as in the $\text{SU}(4)/\text{SU}(3)$ case explained previously. For generic p, q , these are

$$\begin{aligned} \theta &\equiv -\frac{3p+q}{4\sqrt{3p^2+q^2}} e^7, & \omega_1 &\equiv \frac{3p(3p+q)}{16(3p^2+q^2)} (e^{12} + e^{34}), & \omega_2 &\equiv \frac{q(3p+q)}{8(3p^2+q^2)} e^{56} \\ \tilde{\omega}^1 &\equiv \frac{3pq(3p+q)^2}{2^8(3p^2+q^2)^2} (e^{1256} + e^{3456}), & \tilde{\omega}^2 &\equiv \frac{9p^2(3p+q)^2}{2^8(3p^2+q^2)^2} e^{1234}, & \tilde{\omega}^0 &\equiv -\frac{9p^2q(3p+q)^3}{2^{11}(3p^2+q^2)^3} e^{123456}, \end{aligned} \quad (\text{E.19})$$

where the numerical coefficients have been chosen for convenience. Further, for $p = q$, there are two LI three-forms α_0, β^0 :

$$\alpha_0 \equiv \frac{3\sqrt{2}}{128} (e^{135} - e^{146} - e^{236} - e^{245}), \quad \beta^0 \equiv -\frac{3\sqrt{2}}{128} (e^{246} - e^{136} - e^{145} - e^{235}). \quad (\text{E.20})$$

We see therefore that for $p \neq q$ the space M^{pq0} does not support LI three-forms and thus no LI $\text{SU}(3)$ -structure. We thus set $p = q$ and, within this class, focus without loss of generality on M^{110} , which does admit a LI $\text{SU}(3)$ -structure. We find then the relation (2.13) with the only non-vanishing coefficient (up to symmetric permutations of the indices)

$$\mathcal{K}_{112} = 2, \quad (\text{E.21})$$

and the following exterior algebra:

$$\begin{aligned} d\theta &= 2(\omega_1 + \omega_2), & d\omega_1 &= d\omega_2 = 0, & d\tilde{\omega}^1 &= d\tilde{\omega}^2 = 0, & d\tilde{\omega}^0 &= 0, \\ d\alpha_0 &= -4\theta \wedge \beta^0, & d\beta^0 &= 4\theta \wedge \alpha_0, \end{aligned} \quad (\text{E.22})$$

leading to the corresponding entries in tables 3 and 4.

For the LI SU(3)-structure, we take the following expansion

$$\eta = e^V \theta, \quad J = e^{2U_1} \omega_1 + e^{2U_2} \omega_2, \quad \Omega = 2e^{2U_1+U_2}(\alpha_0 + i\beta^0), \quad (\text{E.23})$$

where U_1, U_2, V become scalar fields in the effective theory. Another useful parametrization is in terms of the scalars that enter definite supermultiplets, u_1, u_2, ϕ ,

$$u_1 = U_1 + \frac{1}{2}V, \quad u_2 = U_2 + \frac{1}{2}V, \quad \phi = -2U_1 - U_2, \quad (\text{E.24})$$

so that $v^i \equiv e^{2u_i}$, $i = 1, 2 = n_V$, and ϕ is the 4D dilaton. At a generic point (U_1, U_2, V) in LI moduli space, the only active classes of the SU(3)-structure (E.23) are R, E and T_1 . We omit the details here, and refer again to Q^{111} , which we discuss in the next section. This SU(3)-structure leads to the metric

$$ds_7^2 = \frac{3}{16}e^{2U_1} [(e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2] + \frac{1}{8}e^{2U_2} [(e^5)^2 + (e^6)^2] + e^{2V}\theta^2. \quad (\text{E.25})$$

It can be verified that $U_1 = U_2 = V = 0$ is the canonically normalized SE₇ point: see next section for details on the similar Q^{111} case.

Finally, observe that the effect of the three scalars U_1, U_2, V in the metric is to independently rescale the fiber and the factors $\mathbb{CP}^2 \times \mathbb{CP}^1$ in the base. By similar arguments as in the previous subsection, \mathbb{CP}^2 can be replaced by any compact, 4D Kähler–Einstein manifold, leading to exactly the same truncation as M^{110} . The only other regular, compact Kähler–Einstein four-folds are del Pezzo surfaces.

E.3 Q^{111}

Q^{pqr} is the coset G/H with $G = \text{SU}(2)^3$ and $H = \text{U}(1)^2$, and p, q, r describing the embedding of H into G . Using Pauli matrices σ^i , $i = 1, 2, 3$, we write the nine generators of $G = \text{SU}(2)^3$ in the fundamental in the following block diagonal form:

$$T_i = -\frac{i}{2} \begin{pmatrix} \sigma^i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_{3+i} = -\frac{i}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma^i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_{6+i} = -\frac{i}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma^i \end{pmatrix}. \quad (\text{E.26})$$

The seven generators that exponentiate into the coset are

$$\{\mathcal{K}_\alpha\} = \{T_1, T_2, T_4, T_5, T_7, T_8, Z_1 \equiv \frac{1}{\sqrt{p^2+q^2+r^2}}(pT_3 + qT_6 + rT_9)\}, \quad \alpha = 1, \dots, 7, \quad (\text{E.27})$$

while the generators of $H = \text{U}(1)^2$ are taken to be

$$\begin{aligned} \{\mathcal{H}_\sigma\} = \{Z_2 \equiv \frac{1}{\sqrt{p^2+q^2}}(qT_3 - pT_6), \\ Z_3 \equiv \frac{1}{\sqrt{p^2+q^2+r^2}\sqrt{p^2+q^2}}(prT_3 + qrT_6 - (p^2 + q^2)T_9)\}, \quad \sigma = 1, 2. \end{aligned} \quad (\text{E.28})$$

For generic p, q, r , Q^{pqr} has the following LI forms:

$$\begin{aligned}\theta &\equiv -\frac{\sqrt{3}}{4}e^7, & \tilde{\omega}^0 &\equiv -\frac{3\sqrt{3}}{2^9} \frac{pqr}{(p^2+q^2+r^2)^{3/2}} e^{123456} \\ \omega_1 &\equiv \frac{\sqrt{3}}{8} \frac{p}{\sqrt{p^2+q^2+r^2}} e^{12}, & \omega_2 &\equiv \frac{\sqrt{3}}{8} \frac{q}{\sqrt{p^2+q^2+r^2}} e^{34}, & \omega_3 &\equiv \frac{\sqrt{3}}{8} \frac{r}{\sqrt{p^2+q^2+r^2}} e^{56} \\ \tilde{\omega}^1 &\equiv \frac{3}{64} \frac{qr}{p^2+q^2+r^2} e^{3456}, & \tilde{\omega}^2 &\equiv \frac{3}{64} \frac{pr}{p^2+q^2+r^2} e^{1256}, & \tilde{\omega}^3 &\equiv \frac{3}{64} \frac{pq}{p^2+q^2+r^2} e^{1234}\end{aligned}\quad (\text{E.29})$$

where the numerical coefficients have been chosen for convenience. Further, for $p = q = r$, there are two LI three-forms α_0, β^0 :

$$\alpha_0 \equiv \frac{\sqrt{2}}{64} (e^{135} - e^{146} - e^{236} - e^{245}), \quad \beta^0 \equiv -\frac{\sqrt{2}}{64} (e^{246} - e^{136} - e^{145} - e^{235}). \quad (\text{E.30})$$

Just like M^{pq0} , the space Q^{pqr} can only have a LI SU(3)-structure if $p = q = r$, which we thus fix henceforth. The invariant forms (E.29), (E.30) can be seen to satisfy the algebraic constraints (2.7)–(2.9), and the relation (2.13) with

$$\mathcal{K}_{123} = 1 \quad (\text{E.31})$$

as the only non-vanishing coefficient (up to symmetric permutations of the indices). Furthermore we find the following exterior algebra:

$$\begin{aligned}d\theta &= 2(\omega_1 + \omega_2 + \omega_3), & d\omega_1 &= d\omega_2 = d\omega_3 = 0, & d\tilde{\omega}^1 &= d\tilde{\omega}^2 = d\tilde{\omega}^3 = 0, \\ d\tilde{\omega}^0 &= 0, & d\alpha_0 &= -4\theta \wedge \beta^0, & d\beta^0 &= 4\theta \wedge \alpha_0,\end{aligned}\quad (\text{E.32})$$

from which the geometric fluxes are read off.

The coset Q^{111} supports the following LI SU(3)-structure:

$$\eta = e^V \theta, \quad J = e^{2U_1} \omega_1 + e^{2U_2} \omega_2 + e^{2U_3} \omega_3, \quad \Omega = 2e^{U_1+U_2+U_3} (\alpha_0 + i\beta^0), \quad (\text{E.33})$$

where the U_1, U_2, U_3, V become scalar fields in the 4D theory. The alternative parametrization

$$u_1 = U_1 + \frac{1}{2}V, \quad u_2 = U_2 + \frac{1}{2}V, \quad u_3 = U_3 + \frac{1}{2}V, \quad \phi = -U_1 - U_2 - U_3. \quad (\text{E.34})$$

is useful to make contact with supermultiplets: here we have $v^i \equiv e^{2u_i}$, $i = 1, 2, 3 = n_V$, and ϕ the effective 4D dilaton. For this coset we will provide some explicit details on the intrinsic torsion. The only non-vanishing torsion classes of the SU(3)-structure (E.33) are

$$\begin{aligned}R &= \frac{2}{3} e^V (e^{-2U_1} + e^{-2U_2} + e^{-2U_3}), & E &= 4ie^{-V}, \\ T_1 &= \frac{2}{3} e^V \left[(2 - e^{2U_1-2U_2} - e^{2U_1-2U_3}) \omega_1 + (2 - e^{-2U_1+2U_2} - e^{2U_2-2U_3}) \omega_2 \right. \\ &\quad \left. + (2 - e^{-2U_1+2U_3} - e^{-2U_2+2U_3}) \omega_3 \right],\end{aligned}\quad (\text{E.35})$$

in agreement with (2.22).

The LI metric on Q^{111} corresponding to the above LI $SU(3)$ -structure is

$$ds_7^2 = \frac{1}{8}e^{2U_1} [(e^1)^2 + (e^2)^2] + \frac{1}{8}e^{2U_2} [(e^3)^2 + (e^4)^2] + \frac{1}{8}e^{2U_3} [(e^5)^2 + (e^6)^2] + e^{2V}\theta^2. \quad (\text{E.36})$$

The effect of the scalars in the metric (E.36) is to independently rescale the fiber and the three \mathbb{CP}^1 factors in the base. Also, we will present the Ricci tensor corresponding to the metric (E.36). Although we have not specified the general expression of the Ricci tensor in terms of the torsion classes in (2.3), we can nevertheless compute it by means of coset technology. We find the following non-vanishing components for the Ricci tensor,

$$\begin{aligned} R_{11} = R_{22} &= 1 - \frac{1}{4}e^{-2U_1}, & R_{33} = R_{44} &= 1 - \frac{1}{4}e^{-2U_2}, & R_{55} = R_{66} &= 1 - \frac{1}{4}e^{-2U_3}, \\ R_{77} &= \frac{3}{8}e^{4V} (e^{-2U_1} + e^{-2U_2} + e^{-2U_3}). \end{aligned} \quad (\text{E.37})$$

Note that, at the point $U_1 = U_2 = U_3 = V = 0$, the metric becomes Einstein with canonical normalization, $R_{mn} = 6g_{mn}$, and the torsion classes (E.35) become SE_7 , as given in (2.4).

Finally, note that the metric, the Ricci tensor and the torsion classes for M^{110} and $SU(4)/SU(3)$ can be recovered from the Q^{111} ones by setting $U_3 = U_1$, and $U_3 = U_1 = U_2$, respectively.

E.4 $V_{5,2}$

The Stiefel manifold $V_{5,2}$ is the coset G/H with $G = SO(5)$ and $H = SO(3)$ canonically embedded into $SO(5)$, i.e. the embedding is such that $\mathbf{5} \rightarrow \mathbf{4} + \mathbf{1} \rightarrow \mathbf{3} + \mathbf{1} + \mathbf{1}$ under $SO(5) \supset SO(4) \supset SO(3)$. The generators $M_{ij} = -M_{ji}$, $i, j = 1, \dots, 5$, in the fundamental of $SO(5)$ can be taken to be $(M_{ij})^m{}_n \equiv 2\delta_{[i}^m\delta_{j]}^n$, with

$$\{\mathcal{K}_\alpha\} = \{M_{14}, M_{15}, M_{24}, M_{25}, M_{34}, M_{35}, M_{45}\}, \quad \alpha = 1, \dots, 7, \quad (\text{E.38})$$

exponentiating into the coset, while

$$\{\mathcal{H}_\sigma\} = \{M_{12}, M_{13}, M_{23}\}, \quad \sigma = 1, 2, 3, \quad (\text{E.39})$$

define $H = SO(3)$.

We find the following LI forms for $V_{5,2}$:

$$\begin{aligned} \theta &\equiv \frac{3}{4}e^7, \\ \omega_1 &\equiv \frac{3}{8}(e^{12} + e^{34} + e^{56}), \quad \tilde{\omega}^1 \equiv \frac{3}{64}(e^{1234} + e^{1256} + e^{3456}), \quad \tilde{\omega}^0 \equiv -\frac{27}{29}e^{123456} \\ \alpha_0 &\equiv \frac{3\sqrt{3}}{16\sqrt{2}}e^{135}, \quad \beta^0 \equiv -\frac{3\sqrt{3}}{16\sqrt{2}}e^{246}, \\ \alpha_1 &\equiv \frac{3}{16\sqrt{2}}(e^{136} + e^{145} + e^{235}), \quad \beta^1 \equiv \frac{3}{16\sqrt{2}}(e^{146} + e^{236} + e^{245}). \end{aligned} \quad (\text{E.40})$$

They satisfy the algebraic constraints (2.7)–(2.9), and the relation (2.13) with

$$\mathcal{K}_{111} = 6. \quad (\text{E.41})$$

Furthermore, they close into the following exterior algebra

$$\begin{aligned} d\theta &= 2\omega, & d\omega_1 &= 0, & d\tilde{\omega}^1 &= 0, & d\tilde{\omega}^0 &= 0, \\ d\alpha_0 &= -\frac{4}{\sqrt{3}}\theta \wedge \alpha_1, & d\beta^0 &= -\frac{4}{\sqrt{3}}\theta \wedge \beta^1, \\ d\alpha_1 &= \frac{4}{\sqrt{3}}\theta \wedge \alpha_0 - \frac{8}{3}\theta \wedge \beta^1, & d\beta^1 &= \frac{8}{3}\theta \wedge \alpha_1 + \frac{4}{\sqrt{3}}\theta \wedge \beta^0, \end{aligned} \quad (\text{E.42})$$

from which we can read off the geometric fluxes presented in the tables 3 and 4.

For this model $n_H = 2$ so that the moduli space of almost complex deformations is non-trivial. Using the algorithm of appendix B one straightforwardly finds the prepotential associated to the basis of (α, β) -forms of (E.42):

$$\mathcal{G} = -\frac{1}{3\sqrt{3}} \frac{(Z^1)^3}{Z^0}. \quad (\text{E.43})$$

It is well-known that the c-map for this prepotential leads to the quaternionic-Kähler manifold $\frac{G_{2(2)}}{SO(4)}$.

We can now parameterize the LI $SU(3)$ -structure as follows:

$$\eta = e^V \theta, \quad J = e^{2U_1} \omega_1, \quad \Omega = e^{3U_1+3\varphi} \left(\alpha_0 + \sqrt{3}z\alpha_1 + \sqrt{3}z^2\beta^1 - z^3\beta^0 \right), \quad (\text{E.44})$$

where we introduced the 4D scalar fields V, U_1 and

$$z = \frac{Z^1}{\sqrt{3}Z^0} = \chi + ie^{-2\varphi}. \quad (\text{E.45})$$

As an alternative to the first two fields it is sometimes useful to introduce

$$u_1 = U_1 + \frac{1}{2}V, \quad \phi = -3U_1, \quad (\text{E.46})$$

where $v^1 \equiv e^{2u_1}$ and ϕ is the 4D dilaton. The LI metric associated to this $SU(3)$ -structure takes the following block-form in the e^α -basis:

$$g = \frac{3}{8} e^{2U_1} \begin{pmatrix} L^T L & 0 & 0 & 0 \\ 0 & L^T L & 0 & 0 \\ 0 & 0 & L^T L & 0 \\ 0 & 0 & 0 & \frac{3}{2}e^V \end{pmatrix}, \quad \text{with} \quad L = \begin{pmatrix} e^\varphi & 0 \\ \chi & e^{-\varphi} \end{pmatrix}. \quad (\text{E.47})$$

The non-vanishing torsion classes of the $SU(3)$ -structure (E.44), in terms of the parameters $(U_1, z = \chi + ie^{-2\varphi}, V)$ and the LI forms (E.40) are:

$$\begin{aligned} R &= 2e^{-2U_1+V}, & E &= -4e^{-V} \frac{(1+|z|^2)}{(z-\bar{z})}, \\ S &= e^{3U_1+3\varphi-V} \left[(4z - e^V E) \alpha_0 + \frac{1}{\sqrt{3}} (4(2z^2 - 1) - 3ze^V E) \alpha_1 \right. \\ &\quad \left. + z^2 (4 + ze^V E) \beta^0 + \frac{1}{\sqrt{3}} z (4z^2 - 8 - 3ze^V E) \beta^1 \right]. \end{aligned} \quad (\text{E.48})$$

For $U_1 = \varphi = \chi = V = 0$, these become SE_7 torsion classes, as given in (2.4), and the metric accordingly becomes Einstein, with canonical normalization $R_{mn} = 6 g_{mn}$.

Finally, we remark that we find the Sasaki–Einstein truncation of section E.1 as a subtruncation by keeping only the following linear combinations of the three-forms:

$$(\alpha'_0, \beta'^0) = (\alpha_0 - \sqrt{3}\beta^1, \beta^0 + \sqrt{3}\alpha_1), \quad (\text{E.49})$$

which amounts to putting $z = i$ or $\chi = \varphi = 0$.

E.5 $S^7 = \text{Sp}(2)/\text{Sp}(1)$

The seven-sphere considered as the coset space $\text{Sp}(2)/\text{Sp}(1)$ actually supports a tri-Sasakian structure. The reduced theory is equivalent to the 4D, $N = 4$ supergravity obtained from the universal truncation on any tri-Sasaki space [13, 45]. In line with our present $N = 2$ interest, here we will determine the largest $N = 2$ truncation of the universal $N = 4$ theory. It will turn out that we can find a set of expansion forms obeying the constraints of section 2 by imposing invariance under a certain discrete symmetry group on the expansion forms of the $N = 4$ reduction. Just like the tri-Sasakian reduction this $N = 2$ reduction will be valid for a generic tri-Sasakian manifold.

As in section E.4, we take the generators $M_{ij} = -M_{ji}$, $i, j = 1, \dots, 5$ of $\mathfrak{sp}(2) \approx \mathfrak{so}(5)$ to be $(M_{ij})^m_n \equiv 2\delta^m_{[i}\delta_{j]n}$. Introducing the following $\mathfrak{sp}(1) + \mathfrak{sp}(1)$ generators $J_1, J_2, J_3, L_1, L_2, L_3$,

$$\begin{aligned} J_1 &= \frac{1}{2}(M_{23} + M_{14}), & J_2 &= \frac{1}{2}(-M_{13} + M_{24}), & J_3 &= \frac{1}{2}(M_{12} + M_{34}), \\ L_1 &= \frac{1}{2}(M_{23} - M_{14}), & L_2 &= \frac{1}{2}(-M_{13} - M_{24}), & L_3 &= \frac{1}{2}(M_{12} - M_{34}), \end{aligned} \quad (\text{E.50})$$

the generators that define the coset are then

$$\{\mathcal{K}_\alpha\} = \{M_{15}, M_{25}, M_{35}, M_{45}, J_1, J_2, J_3\}, \quad \alpha = 1, \dots, 7, \quad (\text{E.51})$$

while

$$\{\mathcal{H}_\sigma\} = \{L_1, L_2, L_3\}, \quad \sigma = 1, 2, 3. \quad (\text{E.52})$$

define $H = \text{Sp}(1)$.

The LI forms are

$$\begin{aligned} \{\eta^I\} &= \left\{ \frac{1}{\sqrt{2}}e^6, \frac{1}{\sqrt{2}}e^5, \frac{1}{\sqrt{2}}e^7 \right\}, \\ \{J^I\} &= \left\{ \frac{1}{4}(-e^{13} + e^{24}), \frac{1}{4}(e^{14} + e^{23}), \frac{1}{4}(e^{12} + e^{34}) \right\}, \quad I = 1, 2, 3, \end{aligned} \quad (\text{E.53})$$

and wedge product thereof. These form a LI tri-Sasakian structure, and an expansion in these forms leads to the $N = 4$ theory of [13]. We will now construct the largest subset of forms, which obeys the constraints of section 2, and thus leads to an $N = 2$ theory. We start by choosing a direction and singling out $\theta = \eta^3$ (choosing any other direction leads

to an equivalent theory). It turns out that we can find the rest of the expansion forms by imposing a discrete \mathbb{Z}_2 -symmetry, which reads in the e^α -basis

$$e^1 \rightarrow e^1, \quad e^2 \rightarrow e^2, \quad e^3 \rightarrow -e^3, \quad e^4 \rightarrow -e^4, \quad e^5 \rightarrow -e^5, \quad e^6 \rightarrow -e^6, \quad e^7 \rightarrow e^7. \quad (\text{E.54})$$

It follows that the subtruncation based on the surviving expansion forms is consistent. In fact, this discrete symmetry can be expressed entirely in terms of the tri-Sasakian structure as follows

$$\eta^1 \rightarrow -\eta^1, \quad \eta^2 \rightarrow -\eta^2, \quad \eta^3 \rightarrow \eta^3, \quad J^1 \rightarrow -J^1, \quad J^2 \rightarrow -J^2, \quad J^3 \rightarrow J^3, \quad (\text{E.55})$$

which implies it can be imposed for a generic tri-Sasakian manifold.

In the end we take the following set of expansion forms

$$\begin{aligned} \theta &\equiv \frac{1}{\sqrt{2}} e^7, \\ \omega_1 &\equiv \frac{1}{4} (e^{12} + e^{34}), \quad \omega_2 \equiv \frac{1}{2} e^{56}, \quad \tilde{\omega}^1 \equiv \frac{1}{16} (e^{1256} + e^{3456}), \quad \tilde{\omega}^2 \equiv \frac{1}{16} e^{1234}, \quad \tilde{\omega}^0 = -\frac{1}{32} e^{123456} \\ \alpha_0 &\equiv \frac{1}{8\sqrt{2}} (e^{135} - e^{245}) - \frac{1}{8\sqrt{2}} (e^{146} + e^{236}), \quad \beta^0 \equiv \frac{1}{8\sqrt{2}} (e^{136} - e^{246}) + \frac{1}{8\sqrt{2}} (e^{145} + e^{235}), \\ \alpha_1 &\equiv \frac{1}{8\sqrt{2}} (e^{135} - e^{245}) + \frac{1}{8\sqrt{2}} (e^{146} + e^{236}), \quad \beta^1 \equiv \frac{1}{8\sqrt{2}} (e^{136} - e^{246}) - \frac{1}{8\sqrt{2}} (e^{145} + e^{235}), \end{aligned} \quad (\text{E.56})$$

which obey the algebraic constraints (2.7)–(2.9) (with $n_H = n_V = 2$) and (2.13) with the only non-vanishing coefficient (up to permutations of the indices)

$$\mathcal{K}_{112} = 2. \quad (\text{E.57})$$

Furthermore, they close into the differential algebra

$$\begin{aligned} d\theta &= 2(\omega_1 + \omega_2), \quad d\omega_1 = -4\alpha_1, \quad d\omega_2 = 4\alpha_1, \quad d\tilde{\omega}^1 = d\tilde{\omega}^2 = 0, \quad d\tilde{\omega}^0 = 0, \\ d\alpha_0 &= -4\theta \wedge \beta^0, \quad d\beta^0 = 4\theta \wedge \alpha_0, \quad d\alpha_1 = 0, \quad d\beta^1 = 4(\tilde{\omega}^1 - \tilde{\omega}^2), \end{aligned} \quad (\text{E.58})$$

from which as usual we read off the geometric fluxes.

Since $n_H = 2$ the moduli space of almost complex structure is non-trivial. Using the algorithm of appendix B we find the prepotential

$$\mathcal{G} = -\frac{i}{2} ((Z^0)^2 - (Z^1)^2). \quad (\text{E.59})$$

According to the results of [73], the resulting special quaternionic Kähler manifold is $\text{SO}(4,2)/(\text{SO}(4) \times \text{SO}(2))$.

We can now parameterize the LI $\text{SU}(3)$ -structure as follows:

$$\begin{aligned} \eta &= e^V \theta, \quad J = e^{2U_1} \omega_1 + \epsilon e^{V_1+V_2} \omega_2, \\ \Omega &= e^{2U_1+V_1} [(1-iz)\alpha_0 + (1+iz)\alpha_1 + (i+z)\beta^0 + (-i+z)\beta^1], \end{aligned} \quad (\text{E.60})$$

where we introduced a sign $\epsilon = \pm 1$, the 4D scalar fields $V, U_1, V_1 + V_2$ and the almost complex structure modulus

$$z = \chi + i\epsilon e^\varphi, \quad (\text{E.61})$$

with $\varphi \equiv V_2 - V_1$. The choice of ϵ corresponds to two different branches of the $N = 2$ model, as we explain in more detail in section 5.3. As an alternative to these fields it is sometimes useful to introduce

$$\begin{aligned} u_1 &= U_1 + \tfrac{1}{2}V, & u_2 &= \tfrac{1}{2}(V_1 + V_2 + V), \\ \phi &= -2U_1 - \tfrac{1}{2}(V_1 + V_2), & \varphi &= V_2 - V_1. \end{aligned} \quad (\text{E.62})$$

where $v^i \equiv e^{2u_i}$, $i = 1, 2$ and ϕ is the 4D dilaton.

The LI metric associated to this $\text{SU}(3)$ -structure is given by:

$$ds_7^2 = \tfrac{1}{4} \left((e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 \right) + \tfrac{1}{2} e^{2V_1} \left((e^5)^2 + 2\chi e^5 e^6 + (\chi^2 + e^{2\varphi})(e^6)^2 \right) + e^{2V} \theta^2. \quad (\text{E.63})$$

We conclude our description of this coset by explicitly giving some relevant torsion classes of the $\text{SU}(3)$ -structure (E.60):

$$\begin{aligned} R &= \tfrac{2}{3} \left(2e^{V-2U_1} + \epsilon e^{V-V_1-V_2} \right), \\ E &= i e^{-V-V_1-V_2} \left(e^{2V_2} + 2e^{V_1+V_2} + e^{2V_1} (1 + \chi^2) \right), \\ W_1 &= \tfrac{2}{3} (z - i) \left(-e^{V_1-2U_1} + \epsilon e^{-V_2} \right), \\ T_1 &= \tfrac{2}{3} e^V \left[(1 - \epsilon e^{2U_1-V_1-V_2}) \omega_1 + 2(1 - \epsilon e^{-2U_1+V_1+V_2}) \omega_2 \right], \\ W_2 &= \tfrac{2}{3} (z - i) e^{V_1} \left[- (2 + \epsilon e^{2U_1-V_1-V_2}) \omega_1 + 2(1 + 2\epsilon e^{-2U_1+V_1+V_2}) \omega_2 \right], \end{aligned} \quad (\text{E.64})$$

and we omitted the complicated expressions for W_3 and S .

We find two important points in LI moduli space. The first one is $U_1 = U_2 = V_1 = V_2 = V = \chi = 0$ (the round point), where the metric defined by (E.63) becomes Einstein. For $\epsilon = +1$ the torsion classes obey the conditions for a Sasaki-Einstein structure as given in (2.4), while for $\epsilon = -1$ they obey the conditions of (2.5). This means that the supersymmetry of the round solution, which is $N = 3$ in the $N = 4$ theory, is broken to respectively $N = 2$ and $N = 1$ for the branches $\epsilon = \pm 1$.

The other relevant point is $e^{2V_1} = e^{2V_2} = e^{2V} = \tfrac{1}{5} e^{2U_1}$ (the squashed point), which is also an Einstein point for the metric defined by (E.63). At this point, and for $\epsilon = -1$ only, the classes (E.64) become as given in (2.5), thus giving $N = 1$ at this vacuum. In the branch with $\epsilon = 1$ the supersymmetry of this vacuum becomes invisible.

E.6 $N(k, l)$

We finally consider the Aloff–Wallach spaces $N(k, l) = \text{SU}(3)/\text{U}(1)$, introduced in the supergravity literature in [92] (where they were called N^{pqr}). The integers k, l specify the embedding of $\text{U}(1)$ into $\text{SU}(3)$ according to

$$e^{i\theta} \rightarrow \text{diag}(e^{ik\theta}, e^{il\theta}, e^{-i(k+l)\theta}). \quad (\text{E.65})$$

Accordingly, we chose the coset generators to be

$$\{\mathcal{K}_\alpha\} = \left\{ \frac{1}{2i} \left(\lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_1, \lambda_2, \frac{1}{2\sqrt{k^2+kl+l^2}} \left(\sqrt{3}(l+k)\lambda_3 + (l-k)\lambda_8 \right) \right) \right\}, \quad (\text{E.66})$$

in terms of the Gell-Mann matrices λ_a , $a = 1, \dots, 8$, and the generator of $H = \text{U}(1)$ to be

$$\mathcal{H}_1 = \frac{1}{4i\sqrt{k^2+kl+l^2}} \left((l-k)\lambda_3 - \sqrt{3}(l+k)\lambda_8 \right). \quad (\text{E.67})$$

The integers k, l are taken relatively prime, which guarantees the space to be simply connected. Moreover, from (E.65) we see that there is a permutation symmetry $(k, l) \rightarrow (l, k)$ and $(k, l) \rightarrow (k, -k-l)$.

For generic k, l , we chose the LI forms as follows

$$\begin{aligned} \theta &\equiv \frac{\sqrt{k^2+kl+l^2}}{\sqrt{3}} e^7, \\ \omega_1 &\equiv e^{12}, \quad \omega_2 \equiv e^{34}, \quad \omega_3 \equiv e^{56}, \quad \tilde{\omega}^1 \equiv e^{3456}, \quad \tilde{\omega}^2 \equiv e^{1256}, \quad \tilde{\omega}^3 \equiv e^{1234}, \quad \tilde{\omega}^0 \equiv e^{123456}, \\ \alpha_0 &\equiv \frac{1}{2} (e^{135} - e^{146} - e^{236} - e^{245}), \quad \beta^0 \equiv \frac{1}{2} (e^{246} - e^{136} - e^{145} - e^{235}), \end{aligned} \quad (\text{E.68})$$

leading to a truncation with $n_V = 3$, $n_H = 1$. We find furthermore for the tensor \mathcal{K}_{ijk} in (2.13)

$$\mathcal{K}_{123} = 1. \quad (\text{E.69})$$

These LI forms were also found in [42], but that reference seems to have missed the fact that, just like in the case of M^{pqr} and Q^{pqr} , for certain values of k, l an enhancement in the number of LI forms occurs. We find two special cases. For $k = l = 1$ we find $n_V = 3$, $n_H = 2$, while for $k = -l = 1$ we find $n_V = 5$ and $n_H = 2$. We will omit the details for $N(1, 1)$, which leads to an effective theory consisting of the mere addition of a Betti vector multiplet to the $\text{Sp}(2)/\text{Sp}(1)$ model discussed in section E.5, and instead give some details for $N(1, -1)$. We find then two extra two-forms (and corresponding dual four-forms)

$$\begin{aligned} \omega_4 &\equiv (e^{13} + e^{24}), \quad \omega_5 \equiv (e^{14} - e^{23}), \\ \tilde{\omega}^4 &\equiv -\frac{1}{2} (e^{13} + e^{24}) \wedge e^{56}, \quad \tilde{\omega}^5 \equiv -\frac{1}{2} (e^{14} - e^{23}) \wedge e^{56}. \end{aligned} \quad (\text{E.70})$$

The LI forms on $N(k, l)$ for generic k, l obey the algebra

$$\begin{aligned} d\theta &= \frac{1}{2} [l\omega_1 + k\omega_2 - (k+l)\omega_3], \quad d\omega_1 = d\omega_2 = d\omega_3 = \alpha_0, \\ d\alpha_0 &= 0, \quad d\beta^0 = -\tilde{\omega}^1 - \tilde{\omega}^2 - \tilde{\omega}^3, \quad d\tilde{\omega}^1 = d\tilde{\omega}^2 = d\tilde{\omega}^3 = 0, \quad d\tilde{\omega}^0 = 0, \end{aligned} \quad (\text{E.71})$$

giving as usual directly the values for the geometric fluxes.

For $N(1, -1)$ the extra forms give rise to the additional exterior algebra

$$\begin{aligned} d\omega_4 &= -3\theta \wedge \omega_5, \quad d\omega_5 = 3\theta \wedge \omega_4, \\ d\tilde{\omega}_4 &= -3\theta \wedge \tilde{\omega}_5, \quad d\tilde{\omega}_5 = 3\theta \wedge \tilde{\omega}_4, \end{aligned} \quad (\text{E.72})$$

leading to q -charges and thus to vector multiplet gauging, according to the general discussion. In our conventions, the only non-vanishing components for the tensor \mathcal{K}_{ijk} in (2.13) for $N(1, -1)$ are (up to symmetric permutations)

$$\mathcal{K}_{123} = 1, \quad \mathcal{K}_{344} = -2, \quad \mathcal{K}_{355} = -2. \quad (\text{E.73})$$

The space $N(1, -1)$ supports an $\text{SU}(3)$ -structure specified by

$$\eta = e^V \theta, \quad J = e^{-V} v^i \omega_i, \quad \Omega = 2 e^{U_1+U_2+U_3} (\alpha_0 + i\beta^0), \quad (\text{E.74})$$

with

$$\begin{aligned} v^1 &= e^{2U_1+V} \cosh t, & v^2 &= e^{2U_2+V} \cosh t, & v^3 &= e^{2U_3+V}, \\ v^4 &= -e^{U_1+U_2+V} \sinh t \sin \sigma, & v^5 &= e^{U_1+U_2+V} \sinh t \cos \sigma, \\ \phi &= -U_1 - U_2 - U_3. \end{aligned} \quad (\text{E.75})$$

The LI metric is then given by

$$\begin{aligned} ds_7^2 &= e^{2U_1} \cosh t ((e^1)^2 + (e^2)^2) + e^{2U_2} \cosh t ((e^3)^2 + (e^4)^2) \\ &\quad + 2 e^{U_1+U_2} \sinh t (\cos \sigma (e^1 e^3 + e^2 e^4) + \sin \sigma (e^1 e^4 - e^2 e^3)) \\ &\quad + e^{2U_3} ((e^5)^2 + (e^6)^2) + e^V \theta^2. \end{aligned} \quad (\text{E.76})$$

The two (supersymmetric) Einstein metrics supported by any member in the $N(k, l)$ family are also found on $N(1, -1)$, but in this case they become isometric [93], since they are related by an exchange of the values of U_1 and U_2 , which is a symmetry of the coset. The supersymmetric Einstein point is located at

$$\begin{aligned} e^{2U_1} &= \frac{1}{30} (5 \pm \sqrt{5}), & e^{2U_2} &= \frac{1}{30} (5 \mp \sqrt{5}), & e^{2U_3} &= \frac{1}{5}, & e^{2V} &= \frac{16}{45}, \\ t &= 0, & \theta &\text{arbitrary} \end{aligned} \quad (\text{E.77})$$

normalized as $R_{mn} = 6g_{mn}$. In addition, we numerically find an Einstein locus with $t \neq 0$. Because of this non-diagonal entry, this solution was missed in the old supergravity literature, where only diagonal LI metric deformations associated with rescalings of the canonical vielbeins were considered. This Einstein metric is the same as the one found in the mathematical literature in [82], and is analogous to the (non-supersymmetric) Einstein metric on the 5D $T^{1,1}$ coset space found in [94, 10]. In [82], a search for Einstein metrics taking into account the off-diagonal left-invariant modes on $N(1, 1)$ and $N(1, -1)$ was performed, with the conclusion that both these manifolds admit exactly two inequivalent homogeneous Einstein metrics. Hence on $N(k, l)$ there are exactly two inequivalent Einstein metrics for any choice of k and l .

F Supersymmetric spectra

Here we give the spectrum for each model at all supersymmetric, $N = 2$ or $N = 1$, points, and arrange it into representations of $\text{Osp}(4|N)$. The full, untruncated, Kaluza-Klein spectrum of 11D supergravity on M^{110} and Q^{111} has been given in [95] and [96], respectively. We find perfect agreement with those references. Besides the long vector multiplet of the universal truncation, the extra modes in these models arrange themselves into one and two massless (Betti) vector multiplets, respectively. Similarly, the full $V_{5,2}$ spectrum has been given in [97]. As noted there, in this case, multiplet shortening can occur for multiplets such that $\Delta = |R|$, where Δ is the conformal dimension and R the R-charge of the lowest component. We do find this shortening occurring, producing two chiral multiplets in the spectrum besides the universal long vector multiplet. We find a small disagreement with [97] in that our chiral multiplets are, by construction, singlets under $\text{SO}(5)$, rather than lying on a non-trivial representation. We also present the spectra for the $N = 2$ and $N = 1$ points of the squashed S^7 and $N(1, 1)$ models with, respectively, $\epsilon = +1$ and $\epsilon = -1$ in the notation of appendix E.5. Further, we present the (numerical, for some states) spectrum for a representative $(k = 2, l = 1)$, of the generic $N(k, l)$ model, for both $N = 1$ points I and II. We conclude with the (again numerical for some states) $N(1, -1)$ spectrum at the only $N = 1$ point. We believe that these $N(k, l)$ and $N(1, -1)$ spectra are new.

In the tables, we have always chosen the largest root, Δ_+ , of $\Delta(\Delta - 3) = m^2 L^2$, where L is the radius of AdS. When the smallest root Δ_- is also needed to fill out a multiplet, we have denoted the corresponding entry as (Δ_+, Δ_-) . Besides the supersymmetric spectra, we also include the non-supersymmetric spectra at the corresponding skew-whiffed points. In our conventions, these are attained for $e_0 > 0$ and $e_0 < 0$, respectively. We have also included the vector spectrum. To compute the vector masses, we have chosen the frame of the action of section 3.2, as naturally given by the dimensional reduction. In this frame, the two-form B appears as massive (in fact, it acquires the same mass across all models, for both the $N = 2$ and $N = 1$ points). For both vectors and scalars, we have followed the mass matrix conventions of [5]. The spectrum for $\text{SU}(4)/\text{SU}(3)$ coincides with the one given in that reference.

Finally, note that some of the subtruncation patterns discussed in section 5 can be appreciated from the tables.

spin	$S^7 = \frac{SU(4)}{SU(3)}$ mass eigenstate	M^{110} mass eigenstate	Q^{111} mass eigenstate	$V_{5,2}$ mass eigenstate	$S^7 = \frac{Sp(2)}{Sp(1)}$ mass eigenstate	$N(1, 1)$ mass eigenstate	$U(1)$ R-charge	$m^2 L^2$ ^{susy}	Δ	$m^2 L^2$ ^{skew}	Δ
2	$g_{\mu\nu}$	$g_{\mu\nu}$	$g_{\mu\nu}$	$g_{\mu\nu}$	$g_{\mu\nu}$	$g_{\mu\nu}$	0	0	3	0	3
1	A^0	A^0	A^0	A^0	A^0	A^0	0	0	2	0	2
1	B_2	B_2	B_2	B_2	B_2	B_2	0	12	5	12	5
0	$6U_1 + V$	$4U_1 + 2U_2 + V$	$2U_1 + 2U_2 + 2U_3 + V$	$6U_1 + V$	$4U_1 + V_1 + V_2 + V_3$	$2U_1 + 2U_2 + V_1 + V_2 + V_3$	0	18	6	18	6
0	$3b^1$	$2b^1 + b^2$	$b^1 + b^2 + b^3$	$3b^1$	$2b^1 + b^2$	$b^1 + b^2 + b^3$	0	10	5	-2	2
0	ξ^0	ξ^0	ξ^0	$\xi'^0 \equiv \frac{1}{2}\xi^0 + \frac{\sqrt{3}}{2}\tilde{\xi}_1$	ξ^0	ξ^0	2	10	5	-2	2
0	$\tilde{\xi}_0$	$\tilde{\xi}_0$	$\tilde{\xi}_0$	$\tilde{\xi}'_0 \equiv -\frac{\sqrt{3}}{2}\xi^1 + \frac{1}{2}\tilde{\xi}_0$	$\tilde{\xi}_0$	$\tilde{\xi}_0$	-2	10	5	-2	2
0	$3U_1 - 3V$	$2U_1 + U_2 - 3V$	$U_1 + U_2 + U_3 - 3V$	$3U_1 - 3V$	$2U_1 - 2V_3$	$U_1 + U_2 - 2V_3$	0	4	4	4	4
1	-	-	-	-	A^2	A^2	0	6	4	6	4
0	-	-	-	-	$2b^1 - \tilde{\xi}_1$	$b^1 + b^2 - \tilde{\xi}_1$	0	10	5	-2	2
0	-	-	-	-	$2U_1 - V_1 - V_2$	$U_1 + U_2 - V_1 - V_2$	0	4	4	4	4
0	-	-	-	-	$\varphi \equiv V_2 - V_1$	$\varphi \equiv V_2 - V_1$	2	4	4	4	4
0	-	-	-	-	χ	χ	-2	4	4	4	4
0	-	-	-	-	$2b^1 - 2b^2 + 2\tilde{\xi}_1$	$b^1 + b^2 - 2b^3 + 2\tilde{\xi}_1$	0	0	3	18	6
1	-	A^2	A^2	-	-	A^3	0	0	2	0	2
0	-	$U_1 - U_2$	$U_1 - U_2$	-	-	$U_1 - U_2$	0	-2	(2, 1)	-2	2
0	-	$b_1 - b_2$	$b_1 - b_2$	-	-	$b_1 - b_2$	0	-2	(2, 1)	4	4
1	-	-	A^3	-	-	-	0	0	2	0	2
0	-	-	$U_1 - U_3$	-	-	-	0	-2	(2, 1)	-2	2
0	-	-	$b_1 - b_3$	-	-	-	0	-2	(2, 1)	4	4
0	-	-	-	φ	-	-	$-\frac{8}{3}$	$-\frac{20}{9}$	$\frac{5}{3}$	$-\frac{20}{9}$	$\frac{5}{3}$
0	-	-	-	$\xi'^1 \equiv -\frac{\sqrt{3}}{2}\xi^0 + \frac{1}{2}\tilde{\xi}_1$	-	-	$-\frac{2}{3}$	$-\frac{14}{9}$	$\frac{2}{3}$	$\frac{22}{9}$	$\frac{11}{3}$
0	-	-	-	χ	-	-	$-\frac{4}{3}$	$-\frac{20}{9}$	$\frac{5}{3}$	$-\frac{20}{9}$	$\frac{5}{3}$
0	-	-	-	$\tilde{\xi}'_1 \equiv -\frac{1}{2}\xi^1 - \frac{\sqrt{3}}{2}\tilde{\xi}_0$	-	-	$\frac{2}{3}$	$-\frac{14}{9}$	$\frac{2}{3}$	$\frac{22}{9}$	$\frac{11}{3}$

Table 7: The bosonic LI spectrum on the $N = 2$ and skew-whiffed points across all models. A hyphen indicates an absent state. Horizontal lines separate the different $OSp(4|2)$ multiplets at the susy point. From top to bottom, these are: gravity multiplet, two long vector multiplets, two massless vector multiplets and two chiral multiplets.

spin	$S^7 = \frac{Sp(2)}{Sp(1)}$ mass eigenstate	$N(1, 1)$ mass eigenstate	$m^2 L^2$	Δ	$m^2 L^2$	Δ
2	$g_{\mu\nu}$	$g_{\mu\nu}$	0	3	0	3
1	B_2	B_2	12	5	12	5
1	A^2	A^2	6	4	6	4
0	$4U_1 + V_1 + V_2 + V_3$	$2U_1 + 2U_2 + V_1 + V_2 + V_3$	18	6	18	6
0	$2b^1 - 5b^2 + 2\tilde{\xi}_1$	$b^1 + b^2 - 5b^3 + 2\tilde{\xi}_1$	10	5	-2	2
0	$V_1 + V_2 - 2V_3$	$V_1 + V_2 - 2V_3$	$\frac{52}{9}$	$\frac{13}{3}$	$\frac{52}{9}$	$\frac{13}{3}$
0	$2b^1 - \tilde{\xi}_1$	$b^1 + b^2 - \tilde{\xi}_1$	$\frac{10}{9}$	$\frac{10}{3}$	$\frac{190}{9}$	$\frac{19}{3}$
0	$\varphi \equiv V_2 - V_1$	$\varphi \equiv V_2 - V_1$	$\frac{52}{9}$	$\frac{13}{3}$	$\frac{52}{9}$	$\frac{13}{3}$
0	ξ^0	ξ^0	$\frac{10}{9}$	$\frac{10}{3}$	$\frac{190}{9}$	$\frac{19}{3}$
0	χ	χ	$\frac{52}{9}$	$\frac{13}{3}$	$\frac{52}{9}$	$\frac{13}{3}$
0	$\tilde{\xi}_0$	$\tilde{\xi}_0$	$\frac{10}{9}$	$\frac{10}{3}$	$\frac{190}{9}$	$\frac{19}{3}$
0	$2b^1 + 30b^2 + 2\tilde{\xi}_1$	$b^1 + b^2 + 30b^3 + 2\tilde{\xi}_1$	$-\frac{8}{9}$	$\frac{8}{3}$	$\frac{10}{9}$	$\frac{10}{3}$
0	$6U_1 - 2V_1 - 2V_2 - 2V_3$	$3U_1 + 3U_2 - 2V_1 - 2V_2 - 2V_3$	$-\frac{20}{9}$	$\frac{5}{3}$	$-\frac{20}{9}$	$\frac{5}{3}$
1	A^0	A^0	0	2	0	2
1	-	A^3	0	2	0	2
0	-	$b_1 - b_2$	$\frac{70}{9}$	$\frac{14}{3}$	$-\frac{20}{9}$	$\frac{5}{3}$
0	-	$U_1 - U_2$	$\frac{22}{9}$	$\frac{11}{3}$	$\frac{22}{9}$	$\frac{11}{3}$

Table 8: The bosonic LI spectrum on the squashed ($N = 1$) and skew-whiffed points in the $S^7 = Sp(2)/Sp(1)$ and $N(1, 1)$ models.

spin	$N(2, 1)_I$ mass eigenstate	$m^2 L^2$	Δ	$m^2 L^2$	Δ
2	$g_{\mu\nu}$	0	3	0	3
1	B_2	12	5	12	5
1	A^2	6	4	6	4
0	$2U_1 + 2U_2 + 2U_3 + V$	18	6	18	6
0	$b^1 + b^2 + b^3 - 2\tilde{\xi}_0$	10	5	-2	2
0	$x'_i b^i + \tilde{\xi}_0$	$m^2 L^2$	$\Delta_1 + 1$	$m^2 L^2$	$\Delta_1 - 2$
0	$x^i U_i + V$	$m^2 L^2$	Δ_1	$m^2 L^2$	Δ_1
0	$y^i U_i + V$	$m^2 L^2$	$\Delta_2 + 1$	$m^2 L^2$	$\Delta_2 + 1$
0	$y'_i b^i + \tilde{\xi}_0$	$m^2 L^2$	Δ_2	$m^2 L^2$	$\Delta_2 + 3$
0	$z'_i b^i + \xi_0$	$m^2 L^2$	$(\Delta_3 + 1, 2 - \Delta_3)$	$m^2 L^2$	$(5 - \Delta_3, \Delta_3 - 2)$
0	$z^i U_i + V$	$m^2 L^2$	$(3 - \Delta_3, \Delta_3)$	$m^2 L^2$	$(3 - \Delta_3, \Delta_3)$
1	A_1	0	2	0	2
1	A^3	0	2	0	2

Table 9: The bosonic LI spectrum on the $N = 1$ and skew-whiffed critical point I of the $N(2, 1)$ model. We have defined $(\Delta_1, \Delta_2, \Delta_3) = (3.88744, 3.14403, 1.25659)$, $(x_1, x_2, x_3) = (3.17844, 1.62696, -5.8054)$, $(y_1, y_2, y_3) = (-0.176003, -0.837448, 0.013451)$, $(z_1, z_2, z_3) = (-3.43257, 3.09891, -0.666343)$ and, for $i = 1, 2, 3$, $(x'_i, y'_i, z'_i) = (x^i + 1, y^i + 1, z^i + 1)$. Scalar masses not explicitly indicated are related to their corresponding Δ , to their right, via $m^2 L^2 = \Delta(\Delta - 3)$.

spin	$N(2,1)_{II}$ mass eigenstate	$m^2 L^2$ ^{susy}	Δ	$m^2 L^2$ ^{skew}	Δ
2	$g_{\mu\nu}$	0	3	0	3
1	B_2	12	5	12	5
1	A^2	6	4	6	4
0	$2U_1 + 2U_2 + 2U_3 + V$	18	6	18	6
0	$b^1 + b^2 + b^3 + 2\tilde{\xi}_0$	10	5	-2	2
0	$x'_i b^i - \tilde{\xi}_0$	$m^2 L^2$	$\Delta_1 + 1$	$m^2 L^2$	$\Delta_1 - 2$
0	$x^i U_i + V$	$m^2 L^2$	Δ_1	$m^2 L^2$	Δ_1
0	$y^i U_i + V$	$m^2 L^2$	$\Delta_2 + 1$	$m^2 L^2$	$\Delta_2 + 1$
0	$y'_i b^i - \tilde{\xi}_0$	$m^2 L^2$	Δ_2	$m^2 L^2$	$\Delta_2 + 3$
0	$z'_i b^i - \tilde{\xi}_0$	$m^2 L^2$	$\Delta_3 + 1$	$m^2 L^2$	$\Delta_3 + 2$
0	$z^i U_i + V$	$m^2 L^2$	Δ_3	$m^2 L^2$	Δ_3
1	A_1	0	2	0	2
1	A^3	0	2	0	2

Table 10: The bosonic LI spectrum on the $N = 1$ and skew-whiffed critical point II of the $N(2,1)$ model. We have defined $(\Delta_1, \Delta_2, \Delta_3) = (3.67378, 3.32855, 1.65477)$, $(x_1, x_2, x_3) = (23.5626, -26.065, 1.50232)$, $(y_1, y_2, y_3) = (-0.0118381, 0.00857999, -0.996742)$, $(z_1, z_2, z_3) = (-1.68486, -1.33023, 2.0151)$ and, for $i = 1, 2, 3$, $(x'_i, y'_i, z'_i) = (x^i + 1, y^i + 1, z^i + 1)$. Scalar masses not explicitly indicated are related to their corresponding Δ , to their right, via $m^2 L^2 = \Delta(\Delta - 3)$.

spin	$N(1,-1)$ mass eigenstate	$m^2 L^2$ ^{susy}	Δ	$m^2 L^2$ ^{skew}	Δ
2	$g_{\mu\nu}$	0	3	0	3
1	B_2	12	5	12	5
1	A^2	6	4	6	4
0	$2U_1 + 2U_2 + 2U_3 + V$	18	6	18	6
0	$b^1 + b^2 + b^3 - 2\tilde{\xi}_0$	10	5	-2	2
0	$x'_i b^i + \tilde{\xi}_0$	$m^2 L^2$	$\Delta_1 + 1$	$m^2 L^2$	$\Delta_1 - 2$
0	$x^i U_i + V$	$m^2 L^2$	Δ_1	$m^2 L^2$	Δ_1
0	$y^i U_i + V$	$m^2 L^2$	$\Delta_2 + 1$	$m^2 L^2$	$\Delta_2 + 1$
0	$y'_i b^i + \tilde{\xi}_0$	$m^2 L^2$	Δ_2	$m^2 L^2$	$\Delta_2 + 3$
0	$z'_i b^i + \tilde{\xi}_0$	$m^2 L^2$	$\Delta_3 + 1$	$m^2 L^2$	$5 - \Delta_3$
0	$z^i U_i + V$	$m^2 L^2$	Δ_3	$m^2 L^2$	Δ_3
1	A_1	0	2	0	2
1	A^3	0	2	0	2
1	A^4	$\frac{405}{64}$	$\frac{3}{2} + \frac{\sqrt{421}}{8}$	$\frac{405}{64}$	$\frac{3}{2} + \frac{\sqrt{421}}{8}$
0	t	$\frac{277}{64}$	$\frac{3}{2} + \frac{\sqrt{421}}{8}$	$\frac{277}{64}$	$\frac{3}{2} + \frac{\sqrt{421}}{8}$
1	A^5	$\frac{405}{64}$	$\frac{3}{2} + \frac{\sqrt{421}}{8}$	$\frac{405}{64}$	$\frac{3}{2} + \frac{\sqrt{421}}{8}$
0	σ	$\frac{277}{64}$	$\frac{3}{2} + \frac{\sqrt{421}}{8}$	$\frac{277}{64}$	$\frac{3}{2} + \frac{\sqrt{421}}{8}$

Table 11: The bosonic LI spectrum on the $N = 1$ and skew-whiffed critical point of the $N(1,-1)$ model. We have defined $(\Delta_1, \Delta_2, \Delta_3) = (3.72876, 3.28909, 1.56033)$, $(x_1, x_2, x_3) = (-9.69772, 1.52283, 7.17488)$, $(y_1, y_2, y_3) = (0.0177555, -0.968581, -0.0491748)$, $(z_1, z_2, z_3) = (-1.02824, 2.15395, -2.12571)$ and, for $i = 1, 2, 3$, $(x'_i, y'_i, z'_i) = (x^i + 1, y^i + 1, z^i + 1)$. Scalar masses not explicitly indicated are related to their corresponding Δ , to their right, via $m^2 L^2 = \Delta(\Delta - 3)$. For the susy point, these masses are $m^2 = \mu$, with μ the roots of the cubics $\mu^3 - 24\mu^2 - 56\mu + 2160$ and $\mu^3 - 32\mu^2 - 40\mu + 560$.

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